EE 236
Problem set 11
Solutions

1) The Hamiltonian for a charged particle in an electromagnetic field is

$$H = \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 + e\phi$$

and so we can write the $\vec{E}$ field in any one of several gauges.

For a traveling E field, we can take $\phi = 0$ and $E = -\frac{\partial A}{\partial t}$

so we can see that

$$\hat{H}' = -\frac{e}{m}\vec{A}\cdot\hat{p}$$

and $\hat{H}' = -e\vec{E}\cdot\vec{r} = e\,\phi(\vec{r})$

are equivalent by the choice of gauge. However, what Yariv asks for is essentially to show (in the limited case of wavefunctions in a periodic crystal) that $H = \frac{1}{2m}(\vec{P} - \frac{e}{c}\vec{A}) + e\phi$ is invariant under a gauge transformation. The general case can be shown by the substitution:

$$\psi'(\vec{r},t) \to e^{i\lambda(\vec{r},t)}\psi(\vec{r},t)$$

$$\vec{A}' \to \vec{A} - \vec{\nabla} f(\vec{r},t)$$

$$\phi' \to \phi + \frac{1}{c}\frac{\partial}{\partial t} f(\vec{r},t)$$

$$\text{where } \lambda(\vec{r},t) = \frac{e}{\hbar c} f(\vec{r},t)$$

and because an arbitrary change in phase $(e^{i\lambda(\vec{r},t)})$ of a wavefunction does not change any observables.

In the limited case asked for in this problem, we have ψ as the set of solutions in a periodic potential.

that $H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\phi$ is invarient under a gauge transformation

In a gauge transformation

$$\vec{A}' \to \vec{A} - \frac{\hbar c}{e}\vec{\nabla}\lambda(\vec{r},t)$$

$$\phi' \to \phi + \frac{\hbar}{e}\frac{\partial\lambda(\vec{r},t)}{\partial t}$$

$$\hat{p} \to -i\hbar\bar{\nabla}$$

$$H' = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\hat{A} + \hbar\vec{\nabla}\lambda(r,t)\right)^2 + e\left(\phi + \frac{\hbar}{e}\frac{\partial\lambda(\vec{r},t)}{\partial t}\right)$$

So we have

$$\psi(r,t) = \sum_G a_{\nu K G}\, e^{i(\vec{G}-\vec{K})\cdot\vec{r} - i\omega t}$$

$$H'\psi(r,t) = -\frac{e}{m}\vec{A}\cdot(-i\hbar\nabla)\ \psi(r,t)$$

$$= \sum_G i\hbar\frac{e}{m}\vec{A}\cdot(\vec{G}-\vec{K})\, a_{\nu K G}\, e^{i(\vec{G}-\vec{K})\cdot r - i\omega t}$$

$$\frac{1}{c}\frac{\partial f(\vec{r},\vec{t})}{\partial t} = -e\vec{E}\cdot\vec{r}$$

$$E = E_0 e^{i(K\cdot r-\omega t)}$$

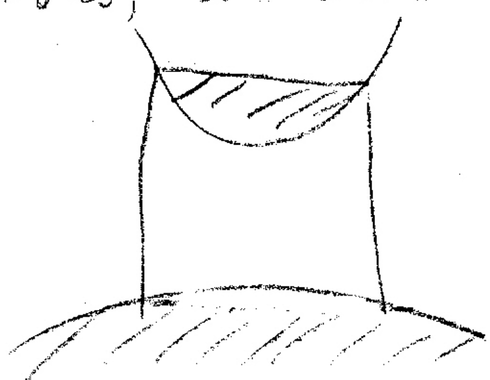
$$f(r,t) = \frac{c}{-i\omega}E_0 e^{i(K\cdot r-\omega t)}$$

$$A = -\frac{c}{-i\omega}\vec{E}_0\cdot\vec{K}\, e^{i(K\cdot r-\omega t)}$$

2) Yariv 11.2

$$\gamma = \frac{\mu^2}{\lambda_0 \varepsilon_0 n \hbar} \left(\frac{2m_r}{\hbar}\right)^{3/2} \left(\omega - E_g/\hbar\right)^{1/2} \cdot \left[f_c(\omega) - f_v(\omega) \right]$$

At 0°K, and equal numbers of electrons and holes, we have the band diagram



$h = 4.134 \times 10^{-15}$ eV/second

$m_{electron} = 5.69 \times 10^{-16}$ eV s^2 cm^{-2}

And we can easily find the $\vec{K}$ vector for the edge of the occupied states by the number of states with $|\vec{K}| < K$

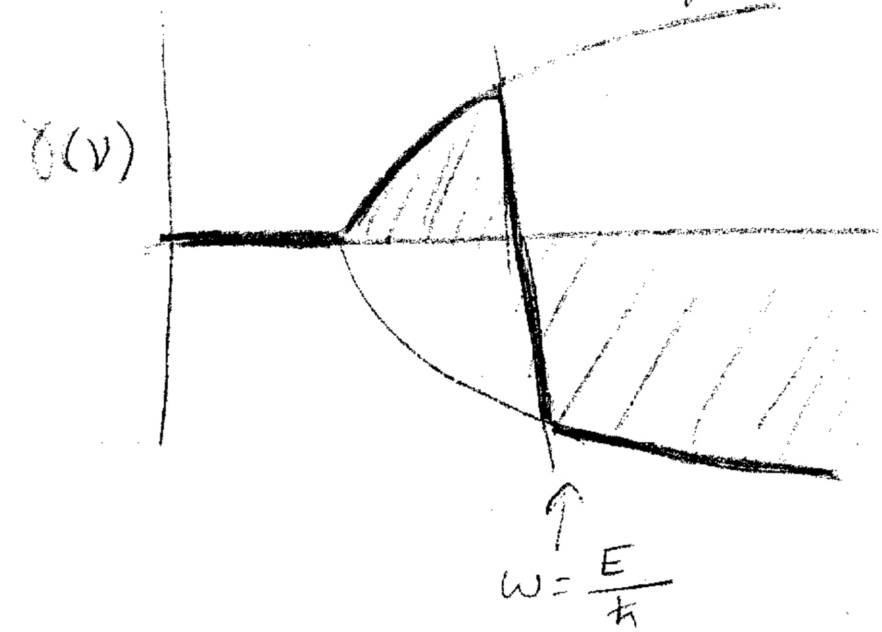
$$\frac{N}{V} = \frac{K^3}{3\pi^2} \qquad \text{(page 99)}$$

$$K = \left(\frac{3\pi^2 N}{V}\right)^{1/3} = 4.4 \times 10^6 \text{ cm}^{-1}$$

The energy corresponding to this $\vec{K}$ vector is $E = E_g + \left(\frac{\hbar^2}{2m_c} + \frac{\hbar^2}{2m_v}\right) K^2$

$$E = (1.45 + .105 + .018) \text{ eV}$$

So we get the gain

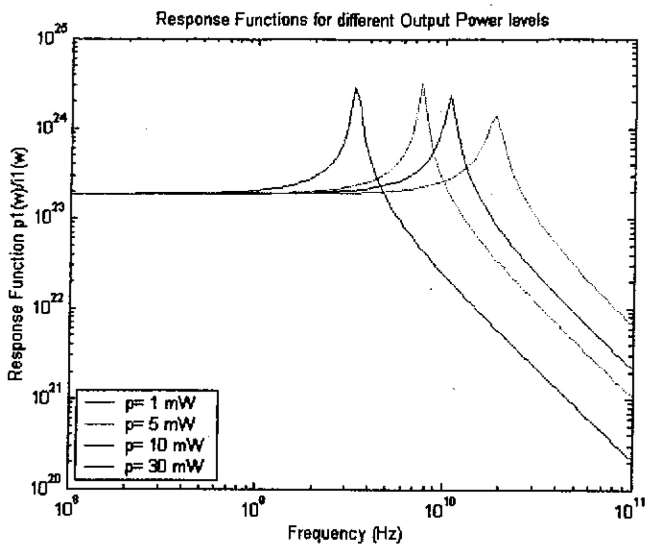


where the energy indicated is the energy from the above calculation

(b) when the temperature is increased, the step function will be slightly smoothed out

(c) T_2 being finite also smooths out the curve, but it also produces a slight smear below the band edge

3) Yariv 11.3



Thanks Linus!

4) Yariv 11.5

$$\frac{dN}{dt} = \frac{I}{eV} - \frac{N}{\tau} - A(N - N_{tr})P$$

$$\frac{dP}{dt} = A(N - N_{tr})P\Gamma_a - \frac{P}{\tau_p}$$

$$N = N_0 + n_1 e^{i\Omega t}$$

$$P = P_0 + p_1 e^{i\Omega t}$$

$$n_1 i\Omega e^{i\Omega t} = \frac{I_0 + i_1 e^{i\Omega t}}{eV}$$

$$- \frac{1}{\tau}(N_0 + n_1 e^{i\Omega t})$$

$$- A(N_0 - N_{tr})(P_0 + P_1 e^{i\Omega t})$$

$$- A n_1 e^{i\Omega t}(P_0 + P_1 e^{i\Omega t})$$

$$p_1 \, i\Omega e^{i\Omega t} = A\,(N_0 - N_{tr})\,\Gamma_a (P_0 + P_1 e^{i\Omega t}) + A\, n_1 e^{i\Omega t}\,\Gamma_0 (P_0 + P_1 e^{i\Omega t}) - \frac{1}{\tau_p}\left(P_0 + p_1 e^{i\Omega t}\right)$$

extracting the terms $\propto e^{i\Omega t}$
(using orthogonality of sinusoidal terms)

$$n_1 \, i\Omega e^{i\Omega t} = \frac{1}{eV}\, i_1 e^{i\Omega t} - \frac{1}{\tau} n_1 e^{i\Omega t} - A(N_1 - N_t)\, P_1 e^{i\Omega t} - A\, n_1 e^{i\Omega t} P_0$$

$$n_1 \, i\Omega = \frac{1}{eV}\, i_1 - \frac{1}{\tau} n_1 - A\,(N_0 - N_t) P_1 - A n_1 P_0$$

$$p_1 \, i\Omega = A\,(N_0 - N_{tr})\,\Gamma_0 \, P_1 + A n_1 \Gamma_0 P_0 - \frac{P_1}{\tau_p}$$

$$p_1 \left\{ i\Omega - A(N_0 - N_{tr})\,\Gamma_0 + \frac{1}{\tau_p} \right\} = A n_1 \Gamma_0 P_0$$

$$P_1 = \frac{A n_1 \Gamma_0 P_0}{\left\{ i\Omega - A(N_0 - N_{tr})\,\Gamma_0 + \frac{1}{\tau_p} \right\}}$$

plugging in p_1

$$n_1 i\Omega = \frac{i_1}{eV} - \frac{n_1}{\tau} - A n_1 P_0$$
$$- A(N_0-N_t)\left\{\frac{A\, n_1 \Gamma_0 P_0}{i\Omega - A(N_0-N_{tr})\Gamma_0 + \frac{1}{\tau_p}}\right\}$$

$$n_1\left\{i\Omega + AP_0 + A(N_0-N_t)\left[\frac{A\Gamma_0 P_0}{i\Omega - A(N_0-N_{tr})\Gamma_0+\frac{1}{\tau_p}}\right]\right\}$$

$$= \frac{i_1}{eV}$$ *since $A(N_0-N_{tr}) = (\Gamma_a \tau_p)^{-1}$

$$n_1 = \frac{\frac{i_1}{eV}\left(i\Omega \cancel{- A(N_0-N_{tr})\Gamma_0 + \frac{1}{\tau_p}}\right)}{(i\Omega+AP_0)\left(i\Omega - A(N_0-N_{tr})\Gamma_0 + \frac{1}{\tau_r}\right) + A(N_0-A_{tr})}$$

$$= i\,\frac{i_1}{eV}\,\frac{\Omega}{(i\Omega)^2 + i\Omega AP_0 + AP_0\, i\Omega - AP_0(N_0-N_{tr})\Gamma_0 + \frac{AP_0}{\tau_r}}$$

$$= -i\,\frac{i_1}{eV}\,\frac{\Omega}{\Omega^2 - \frac{AP_0}{\tau_p} - i\Omega\left(\frac{1}{\tau}+AP_0\right)}$$ (using * again)

(b) $n = n' + i n''$

$$n \equiv \frac{1}{c} \frac{K}{\omega}$$

$$\frac{dP}{dt} = A(N - N_{tr}) P \Gamma_a - \frac{P}{\tau_p}$$

Γ_a - Fill Factor

$$\gamma_{mode}(\nu) = I^{-1} \frac{dI}{dz} \qquad \gamma(\nu) = -\frac{k\chi''}{n^2}$$

(8.2-7)

$$K' \approx k\left[1 + \frac{\chi'(\nu)}{2n^2}\right] - i\,\frac{k\chi''(\nu)}{2n^2}$$

8.2-4

$$n'' = \frac{1}{c\omega} \frac{k\chi''(\nu)}{2n^2} = \frac{1}{c\omega} \frac{1}{2} \gamma(\nu)$$

$$\gamma_{mode}(\nu) = P^{-1} \frac{dP}{dz} = P^{-1} \frac{dP}{dt} \frac{dt}{dz}$$

$$= P^{-1} \frac{n'}{c} \frac{dP}{dt}$$

$$\gamma_{mode}(\nu) = P^{-1} \frac{n'}{c} A(N - N_{tr}) P \Gamma_a$$

$$\gamma(\nu) = \frac{n'}{c} A(N - N_{tr})$$

$$n'' = \frac{1}{c\omega}\,\frac{1}{2}\,\gamma(\nu)$$

$$\gamma(\nu) = \frac{n'}{c} A \,(N - N_{tr})$$

$$2c\omega n'' = \frac{n'}{c} A\,(N - N_{tr})$$

$$2c\omega \frac{\partial n''}{\partial N} = \frac{n'}{c} A$$

$$\boxed{A = \frac{2\omega}{n'}\,\frac{\partial n''}{\partial N}}$$

c)

$$N = N_0 + n_1 e^{i\Omega t}$$

$$\frac{\Delta \nu}{\nu} = -\left(\frac{\Delta n'}{n'}\right) \Gamma_a$$

$$\frac{dn'}{dN} = \alpha \frac{\partial n''}{\partial N}$$

$$\Delta n' = \frac{dn'}{dN} n_1 e^{i\Omega t}$$

$$\Delta n' = \alpha \frac{\partial n''}{dN} n_1 e^{i\Omega t}$$

$$\Delta n' = \alpha n_1 e^{i\Omega t} A \frac{n'}{4\pi\nu}$$

(from part A)

$$\frac{\Delta \nu}{\nu} = -\frac{\Gamma_a \alpha n_1 e^{i\Omega t} A}{4\pi\nu}$$

$$n_1(\Omega) = -i\,\frac{i_1}{eV}\,\frac{\Omega}{\Omega^2 - \frac{AP_0}{\tau_p} - i\Omega\left(\frac{1}{\tau} + AP_0\right)}$$

$$\Omega_R = \sqrt{\frac{AP_0}{\tau_p} - \frac{1}{2}\left(\frac{1}{\tau} + AP_0\right)^2}$$

$$n_1(r) = -i\,\frac{i_1}{eV}\,\frac{\Omega}{\Omega^2 - \Omega_R^2}$$

$$n_1(r) = +i\,\frac{i_1}{eV}\,\frac{\Omega}{\Omega_R^2\left(1 - \Omega^2/\Omega_R^2\right)}$$

$$n_1(r) = i\,\frac{i_1}{eV}\,\frac{\Omega}{\Omega_R^2}$$

$$\frac{\Delta\nu}{\nu} = -\frac{\Gamma_a \alpha A}{4\pi\nu}\,\frac{i_1}{eV}\,\frac{\Omega}{\Omega_R^2}$$

$$\Delta\nu = \frac{i_1(\Omega)}{eV}\left(\frac{\alpha}{4\pi}\right)\frac{\Omega\tau_p}{\Omega_R^2}$$

$$= \frac{i_1(\Omega)}{eV}\left(\frac{\alpha}{4\pi}\right)\frac{\Omega\tau_p}{P_0}$$