Representations, orthogonality

In the first lecture, we defined kets, bras and operators as abstract symbols for dealing with QM systems. Today, we will look at some specific ways of looking at QM problems with a countable number of states.

Example of a QM system with an uncountable number of states: particle in free space \( \mathcal{H} \)

Let's say that there happens to be a set of states such that

\[ |\psi\rangle = A_1 |\phi_1\rangle + A_2 |\phi_2\rangle + A_3 \cdots \]

For some \( A_1 \), \( A_2 \), \( \ldots \)
We then say that the space of all states is spanned by the set \( \{ \Phi_1, \Phi_2, \ldots \} \) and that they form a complete set.

If the set also has the property\(^2\)

\[ \langle \Phi_n | \Phi_m \rangle = 0 \quad \text{if} \quad m \neq n \]

then the set is said to be orthogonal.

If the elements also have the property

\[ \langle \Phi_n | \Phi_n \rangle = 1 \quad \text{for all} \quad n \]

then the set is normalized.

If a complete orthogonal set of \( M \) states spans a space of a \( QM \) system, it is said to be an \( M \)-state system.
Example. If we consider only the spin of an electron, then it is a 2-state system.

\[ \uparrow \quad \downarrow \]

Often, we can approximate a system with many states by limiting the system to just a few of the possibilities. An atom might be modeled using just the "lasing levels" and the photons might be modeled by a ground state & the state with one photon in one mode.

- upper lasing level
- lower lasing level

\[ \uparrow \text{ photon} \]

\[ \text{ground state} \]
If we were to somehow construct a complete orthogonal set of normalized eigenfunctions, we can then use them to translate any problem with states and operators into vectors and matrices

\[
|\Psi\rangle = \sum_n \psi_n |\phi_n\rangle
\]

\[
\langle \phi_m | \Psi \rangle = \sum_n \psi_n \langle \phi_m | \phi_n \rangle
\]

\[
\langle \phi_m | \Psi \rangle = \psi_m
\]

So a general state of the system can be written

\[
|\Psi\rangle = \sum_m \langle \phi_m | \Psi \rangle |\phi_m\rangle
\]

Let's rewrite this as

\[
|\Psi\rangle = \sum_n |\phi_n\rangle \langle \phi_n | \Psi \rangle
\]

We can now identify an expression for an identity operator

\[
I = \sum_n |\phi_n\rangle \langle \phi_n |
\]
and we can use that identity operator to find a matrix representation for any operator \( \hat{A} \):

\[
\sum_m \sum_n \langle \phi_m | \hat{A} | \phi_n \rangle < \phi_n |
\]

Notice that the vectors \( \phi_m \) and the numbers \( A_{mn} \) depend on the representation i.e., exactly which complete, orthonormal set is used to represent them.

Let's look at this from the opposite point of view. If a Q.M. problem can be spanned by a complete, orthogonal set, then any state of the system corresponds to a point in an \( n \) dimensional linear space, and therefore be
represented by a vector, and any operator corresponds to a linear transformation, and can be represented by a matrix.

The results we will obtain are more general, but let's look at just finite linear spaces.

In a Hilbert space, for any operator there are specific states which have the property that

$$\hat{A} |A_n\rangle = a_n |A_n\rangle$$

where $a_n$ is a complex number.

For the case of a finite linear space, this is just the corresponding result for matrices, resulting from any representation.

These specific states are called the eigenstates of the operator $\hat{A}$, the values $a_n$ are the eigenvalues corresponding to those eigenstates.
If we interact with a system or "measure" it, the system in general will change. If we use the operator $\hat{A}$ to represent the change,

If we demand that our measurement be repeatable, and that it tells us something about the state of the system, we will effectively be acting with the operator more than once:

$\hat{A} \mid \psi \rangle \rightarrow \text{measure once}$

$\hat{A} \hat{A} \mid \psi \rangle \rightarrow \text{measure again}$

$\hat{A} \hat{A} \hat{A} \mid \psi \rangle \rightarrow \text{measure a third time}$

We will look at "measurement" more closely later, but for now let's recognize that for a "measurement" to be repeatable, the action of a "measurement" must produce an eigenstate of $\hat{A}$, $\mid \lambda \rangle$, so that further "measurements" don't produce different results each time.
If we look at the Hermitian operators, they have the property which will be nice in this regard, so we might state the theory of "measurement":

1. All observables correspond to Hermitian operators.
2. Results of observations are the eigenvalues of these operators.
3. After a "measurement" the observed system is in the eigenstate corresponding to the observed eigenvalue.

The eigenvalues of a Hermitian operator can be shown to be real:

\[ \hat{A} \mid \phi_n \rangle = \alpha_n \mid \phi_n \rangle \quad \#1 \]
\[ \hat{A} \mid \phi_m \rangle = \alpha_m \mid \phi_m \rangle \quad \#2 \]

because \( \hat{A} \) is Hermitian:

\[ \langle \phi_m | \hat{A} = \alpha^*_m \langle \phi_m | \quad \#3 \]
\[ \langle \phi_n | \hat{A} = \alpha^*_n \langle \phi_n | \quad \#4 \]
we now project \( \phi_m \) on \( \langle \phi_m \rangle \) and use \#3 to project \( \phi_n \rangle \). we get
\[
\langle \phi_m | \hat{A} | \phi_n \rangle = a_n \langle \phi_m | \phi_n \rangle \\
\langle \phi_m | \hat{A}^* | \phi_n \rangle = a_m^* \langle \phi_m | \phi_n \rangle
\]
Subtracting, we get
\[
0 = (a_n - a_m^*) \langle \phi_m | \phi_n \rangle
\]
If \( n = m \) \( a_n \) is real \( (a_n = a_m^*) \)
If \( n \neq m \) and \( a_n \neq a_m^* \) then
\[
\langle \phi_m | \phi_n \rangle = 0 \quad (\text{they are orthogonal})
\]
So the set of all eigenstates from a set of orthogonal states which we can project any state \( \phi \) on \( \hat{A} \) represents... \\
Do they complete? \\
They don't have to be - we must use a set of operators which span the space - so that observing the observable describes the aspects of the N.M. system in a way of \( \hat{A} \).
If an operator which spans the space has $N$ eigenstates, we call that an $N$ state system (2-state system, etc.)

If the eigenstates of an operator $A$ are complete, then any ket in the space can be expressed

$$|\psi\rangle = \sum c_n |\phi_n\rangle$$

where the $c_n$ are complex numbers, that is, that any state $\psi$ can be completely represented by an ordered vector of complex numbers (a column matrix).

Therefore any linear operator can be represented by a matrix.

Any observable will result in a complete set of orthogonal eigenvalues and can be used to represent the same system with different vectors and different matrices.
Let's see how to calculate the \( \hat{B} \)

\[
\langle \Psi \rangle = \sum_n c_n |\Psi_n\rangle \\
\langle a_n | \Psi \rangle = \sum_n \langle a_n | c_n | \Psi_n \rangle \\
\langle a_n | \Psi \rangle = \sum_n c_n \langle a_n | \Psi_n \rangle = c_n \langle a_n | \Psi \rangle \\
c_n = \langle a_n | \Psi \rangle \\
| \Psi \rangle = \sum_n \left[ \sum_m c_m \langle a_m | \Psi \rangle \right] |\Psi_n\rangle \\
| \Psi \rangle = \left[ \sum_n |\Psi_n\rangle \langle a_n | \right] |\Psi\rangle \\
\text{Identity operator - project over the } a_n \text{ and then expand over them} \\
\Rightarrow \text{ we can use these to express arbitrary operators } \hat{B} \text{ as matrices in the } \hat{A} \text{ representation} \\
| \phi \rangle = \hat{B} | \Psi \rangle \\
| \phi \rangle = \sum_n |\Psi_n\rangle \langle a_n | \hat{B} \sum_m c_m |\Psi_m\rangle \langle a_m | \Psi \rangle