Maxwell's equations are the result of Quantum Electrodynamics QED, which is a quantum field theory, but we are going to start at that level and work down to quantum effects. The potentials $V(r)$ described earlier are also derivable from QED.

Because photons interact with each other very very little, Maxwell's equations are accurate to an extremely good approximation in a vacuum.

If we account for all charges and motion of charge explicitly, then we have

\[ \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \]

\[ \nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} \]

\[ \nabla \cdot \vec{B} = 0 \]

\[ \nabla \cdot \vec{E} = \rho_{\text{total}} \]
\[
\vec{D} = \varepsilon_0 \vec{E} + \vec{P}
\]

\[
+ \vec{B} = \mu_0 (\vec{H} + \vec{M})
\]

if we treat all charges explicitly, then \( \vec{P} = 0 \) and we can substitute \( \vec{D} = \varepsilon_0 \vec{E} \) everywhere and not use \( \vec{B} \) as is often done in physics. This set of equations has unique solution for known \( \vec{P}, \vec{E}, \vec{M}, \) and \( \vec{D} \) with reasonable boundary conditions (no unknown inbound radiation field).

However, the fields often affect the motion of charge \( \vec{M} \) as we saw in our perturbation results. This gives us a feedback in our problem.
To make the problem solvable, we will separate the charge into three parts:

Currents in good conductors

\[ \Rightarrow \text{use } BC's \]

Small displacements of bound charge

\[ \Rightarrow \text{use } \vec{P} \Rightarrow \vec{D} \]

Known \( \vec{D} \) (thin wires, charge distribution) can be handled directly.

There are other problems which don't fall into these categories (planar, for example) which need a simultaneous solution for charge and fields.

If we look at a small neutral region of material where \( E(r,t) \) does not vary substantially, we can have a displacement of the average expectation value of the negative charge from that of the positive charge:

\[ \langle \vec{P}_e \rangle - \langle \vec{P}_p \rangle \]
If we define \( \hat{\mathbf{P}} = \mathbf{g} \cdot \mathbf{N} \cdot \hat{\mathbf{r}} \) and we ignore the small scale variations, then we have a current

\[
\hat{\mathbf{j}}_{\text{bound}}(\mathbf{r},t) = \frac{\partial}{\partial t} \hat{\mathbf{P}}(\mathbf{r},t)
\]

\[
+ \hat{\mathbf{P}}_{\text{bound}}(\mathbf{r},t) = -\nabla \cdot \hat{\mathbf{P}}(\mathbf{r},t)
\]

when we then define

\[
\mathbf{D} = \varepsilon_0 \mathbf{E} + \hat{\mathbf{P}}
\]

\[
\nabla \cdot \mathbf{D} = \varepsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot \hat{\mathbf{P}} = \Phi_{\text{total}} - \Phi_{\text{bound}} = \Phi_{\text{free}}
\]

\[
+ \nabla \times \mathbf{H} = \mathbf{j}_{\text{free}} + \frac{\partial \mathbf{D}}{\partial t}
\]

For narrow band solutions, \( \hat{\mathbf{P}} \) is often linearly related to \( \mathbf{E} \), so we will have \( \hat{\mathbf{P}} = \varepsilon \mathbf{K} \mathbf{E} \) where \( \mathbf{K} \) can be a tensor in anisotropic materials, but this lets us break the loop.
We will now study the relationship \( E \rightarrow \mathbf{p} \) and \( \mathbf{p} \rightarrow E \), which will form the core of the interaction of light with matter.

First, note that a charge moving with or against an \( E \) field will add to or take away from the energy of the \( E \) field. The work done in polarizing the QM system is

\[
W = E \cdot \frac{d\mathbf{p}}{dt}
\]

This energy has two important parts: sloshing back and forth from the QM system to the field on every cycle, and a net transfer of energy QM \( \rightarrow \) EM or EM \( \rightarrow \) QM.

If the perturbation is small, the polarization will depend linearly on the \( E \) field:

\[
\mathbf{P}(\mathbf{r}, t) = \int_{-\infty}^{t} E(\mathbf{r}, \tau) h(t-\tau) d\tau
\]

where \( h(t-\tau) \) is the impulse response (tensor).
We can also limit the stimulating field to sinusoidal functional

\[ \vec{E} = R(\vec{E}(r)) e^{i\omega t} \]

where \( \vec{E}(r) \) is now a phasor.

If the relationship between \( \vec{P} \) and \( \vec{E} \) is linear, as explained above, there must be a function \( \chi_0(\omega) \) such that when

\[ \vec{P} = \varepsilon_0 \chi_0 \bar{\vec{E}} \]

\[ + \quad \vec{P}(t) = \text{Re} \left\{ \vec{P}(\tau) e^{-i\omega \tau} \right\} \]

note that \( \vec{P}(\tau) + \bar{\vec{E}}(\tau) \) are complex + independent of time, but the \( \vec{E} \) field + \( \vec{P} \) are both real, unlike the complex \( \psi(\vec{r}, t) \) of Q.M.

This quantity \( \chi_0 \) is called the Electric susceptibility, and is complex
IF we look at the average power from or to the field,

\[
\frac{\text{Power}}{\text{Volume}} = E \cdot \frac{\partial P}{\partial t}
\]

When using phasor notation, if you multiply (for example, to find power) you need to explicitly use the real part

\[
\frac{\text{Power}}{\text{Volume}} = \text{Re}(E e^{i\omega t}) \text{Re}(i\omega P e^{i\omega t})
\]

\[
= \frac{1}{\sqrt{2}} \left( \frac{E e^{i\omega t} + E^* e^{-i\omega t}}{i\omega P e^{i\omega t} - i\omega P^* e^{-i\omega t}} \right)
\]

\[
\text{\textbf{P}} = \varepsilon_0 \chi_e \varepsilon
\]

\[
\frac{\text{Power}}{\text{Volume}} = \frac{1}{\sqrt{2}} \left( E e^{i\omega t} + E^* e^{-i\omega t} \right) \left( i\omega \chi_e \varepsilon e^{i\omega t} - i\omega \chi_e \varepsilon e^{-i\omega t} \right)
\]

\[
= \frac{1}{\sqrt{2}} \left( E (-i\omega \chi_e \varepsilon) \chi_e^* \varepsilon \right) + \frac{1}{\sqrt{2}} \left( i\omega \chi_e \varepsilon \right) E
\]

(Keeping only the D.C. terms)
= \frac{1}{2} \text{Re}(i\omega_0 \chi_e E E^*)

= \frac{\omega}{2} \varepsilon_0 / E^2 \text{Re}(i\chi_e)$