EE 236
9/27/04
0

Maxwell's equations are the result of Quantum Electrodynamics QED which is a quantum field theory, but we are going to start at that level + work down to quantum effects. The potentials $V(\vec{r})$ described earlier are also derivable from QED.

Because photons interact with each other very very little, Maxwell's equations are accurate to an extreemly good approximation, in a vacuum.

If we account for all charges + motions of charge explicitly, then we have

$$\nabla \times \vec{H} = \vec{i} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

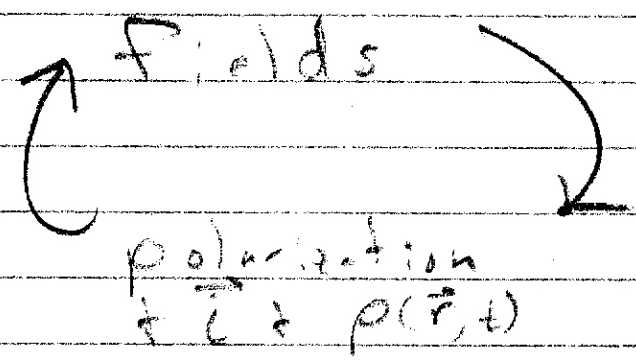
$$\vec{\nabla} \cdot \vec{E} = \rho_{total}$$

where $\vec{D} = \varepsilon_0 E + \vec{P}$

& $\vec{B} = \mu_0(\vec{H} + \vec{M})$

if we treat all charges explicitly, then $\vec{P} = 0$ & we can substitute $\vec{D} = \varepsilon_0 \vec{E}$ everywhere and not use $\vec{D}$ as is often done in physics. This set of equations has unique solutions for known ρ, $\vec{i}$, $\vec{M}$, and $(\vec{P})$ with reasonable boundary conditions (no unknown inbound radiation field)

However the fields often affect the motion of charges & $\vec{M}$, as we saw in our perturbation results. This gives us a feedback in our problem:



To make the problem solvable, we will separate the charge into three parts:

Currents in good conductors
$\Rightarrow$ use B.C.'s

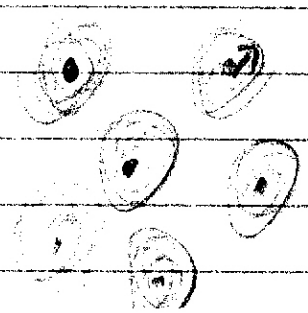
Small displacements of bound charge
$\Rightarrow$ use $\vec{P} \Rightarrow \vec{D}$

Known $\vec{\rho}$, $\vec{j}$ (thin wires, charge distributions) can be handled directly

There are other problems which don't fall into these categories (plasmas for example) which need a simultaneous solution for charges & fields.

If we look at a small neutral region of material where $\vec{E}(r,t)$ does not vary substantially, we can have a displacement of the average expectation value of the negative charge from that of the positive charge

$$\overline{\langle \vec{r}_e \rangle} - \overline{\langle \vec{r}_p \rangle}$$



if we define $\vec{P} = q \cdot N \cdot \overline{\langle \vec{r} \rangle}$
and we ignore the small scale variations, then we have a current

$$\vec{i}_{bound}(\vec{r},t) = \frac{\partial}{\partial t} \vec{P}(\vec{r},t)$$

$$+ \quad \rho_{bound}(\vec{r},t) = -\vec{\nabla} \cdot \vec{P}(r,t)$$

when we then define

$$\vec{D} = \varepsilon_0 E + \vec{P}$$

$$\vec{\nabla} \cdot \vec{D} = \varepsilon_0 \vec{\nabla} \cdot \vec{E} + \vec{\nabla} \cdot \vec{P}$$

$$= \rho_{total} - \rho_{bound} = \rho_{free}$$

$$+ \quad \nabla \times \vec{H} = i_{free} + \frac{\partial \vec{D}}{\partial t}$$

For narrow band solutions, $\vec{P}$ is often linearly related to $\vec{E}$, so we will have $\vec{P} = \varepsilon_0 \chi \vec{E}$ where χ can be a tensor in anisotropic materials, but this lets us break the loop.

We will now study the relationship $\vec{E} \rightarrow \vec{P}$ and $\vec{P} \rightarrow \vec{E}$, which will form the core of the interaction of light with matter.

First, note that a charge moving with or against an $\vec{E}$ field will add to or take away from the energy of the $\vec{E}$ field. The work done in polarizing the Q.M. systems is

$$W = E \cdot \frac{dP}{dt}$$

this energy has two important parts: sloshing back and forth from the QM system to the field on every cycle, and a net transfer of energy $QM \rightarrow EM$ or $EM \rightarrow QM$

If the perturbation is small, the polarization will depend linearly on the E field:

$$\vec{P}(\vec{r},t) = \int_{-\infty}^{t} E(\vec{r},\tau)\, h(t-\tau)\, d\tau$$

where $h(t-\tau)$ is the impulse response (tensor)

We can also limit the stimulating field to sinusoidal functions:

$$\vec{E} = Re\{\underline{\vec{E}(\vec{r})} e^{i\omega t}\}$$

where $\underline{\vec{E}(\vec{r})}$ is now a phasor

if the relationship between $\vec{P}$ & $\vec{E}$ is linear, as explained above, there must be a function $\chi_e(\omega)$ such that when

$$\underline{\vec{P}} = \varepsilon_0 \chi_e \underline{\vec{E}}$$

& $$\vec{P}(t) = Re\{\underline{\vec{P}(\vec{r})} e^{i\omega t}\}$$

note that $\underline{\vec{P}(\vec{r})}$ & $\underline{\vec{E}(\vec{r})}$ are complex & indep. of time, but the $\vec{E}$ field & $\vec{P}$ are both real, unlike the complex $\psi(\vec{r}, t)$ of Q.M.

this quantity χ_e is called the Electric susceptibility, and is complex

If we look at the average power from or to the field,

$$\overline{\frac{\text{power}}{\text{volume}}} = \overline{\vec{E} \cdot \frac{\partial \vec{P}}{\partial t}}$$

When using phasor notation, if you multiply (for example, to find power) you need to explicitly use the real part

$$\frac{\text{Power}}{\text{volume}} = \overline{\underline{E} \cdot \frac{\partial \underline{P}}{\partial t}}$$

$$= \overline{\mathrm{Re}(\underline{E} e^{i\omega t})\, \mathrm{Re}(i\omega \underline{P} e^{i\omega t})}$$

$$= \overline{\tfrac{1}{2}(\underline{E} e^{i\omega t} + \underline{E}^* e^{-i\omega t}) \cdot \tfrac{1}{2}(i\omega \underline{P} e^{i\omega t} - i\omega \underline{P}^* e^{-i\omega t})}$$

$$\underline{\vec{P}} = \varepsilon_0 \chi_e \underline{\vec{E}}$$

$$\frac{\text{Power}}{\text{Volume}} = \tfrac{1}{4}(\underline{\vec{E}} e^{i\omega t} + \underline{\vec{E}}^* e^{-i\omega t})(i\omega\varepsilon_0\chi_e \underline{\vec{E}} e^{i\omega t} - i\omega\varepsilon_0\chi_e^* \underline{\vec{E}}^* e^{i\omega t})$$

$$= \tfrac{1}{4}\left(\underline{\vec{E}}(-i\omega\varepsilon_0)\chi_e^* \underline{\vec{E}} + \underline{\vec{E}}^*(i\omega\varepsilon_0\chi_e)\underline{\vec{E}}\right)$$

(Keeping only the D.C. terms)

$$= \tfrac{1}{2} \mathrm{Re}\left(i\omega_0 \chi_e E E^* \right)$$

$$= \frac{\omega}{2} \varepsilon_0 |E|^2 \mathrm{Re}(i\chi_e)$$