The Kramers-Kronig relationship

When an Electromagnetic Field acts on a population of atoms (matter) it produces a polarization, $P$.

In general, the polarization is a function of the Electric Field locally, but not instantly, $\Rightarrow$ it depends on the history of the Electric Fields in the past.

If the relationship between the electric field and the polarization is linear, i.e.

$$P(t) = E_1(t) + E_2(t) = P_1 E_1(t) + P_2 E_2(t)$$

For any electric fields $E_1(t) + E_2(t)$, we can find the response $P(t)$ to an impulse $E(t) = \delta(t-t')$ and then compose any $E(t)$ as a sum over these delta functions. We get a Green's Function for $P$.
the equation \( P = \int_0^\infty G(t) E(t) e^{-\omega t} dt \)

(for a time invariant, linear & causal relationship)

...is a special case, when transformed as discussed earlier, we have

\[ E(w) = E_0 + \int_0^\infty G(t) e^{-\omega t} dt \]

...and so we have

\[ \overline{E_w} = E_w \overline{E} \]

\[ \Rightarrow E_w \text{ is now restricted to the isotropic, not optically active case} \]

...so it is a complex scalar

The Kramer-Kronig relations are a consequence of these approximations. Yariv treats this in appendix E, or better, see Jackson 3rd ed p332

This follows from the realization that \( E(w) \) is a complex variable of the complex variable \( w \), and is an analytic function of \( w \) in the upper half plane

We also need the requirement \( G(t) \to 0 \) as \( \tau \to \infty \), so the material has no "memory"
Note that the equation

\[ P = \int_{-\infty}^{\infty} q(z) E(z^2) \, dz \]

is not in phasor notation \( \Rightarrow G(z) \)

must be a real function \( (E + P \text{ are real}) \)

So when we look at

\[ E = E_0 + \int_{0}^{\infty} G(z)e^{-iz} \, dz \]

\[ E^* = E_0 + \int_{-\infty}^{0} G(z)e^{-iz} \, dz \]

\[ \Rightarrow E(\omega) = E^*(\omega^*) \]

Note that a pole in the upper half plane corresponds to a complex frequency

\[ \omega = \omega_r + i \omega_i \]

where there is a potentially \( \nabla \) in the presence of a decaying \( E \) field

\[ \vec{E} = E(r) e^{i(\omega_r t - \omega_i r)} \]

\[ = E(r) e^{i\omega_r t - \omega_i r} \]
This is all simply a consequence of the fact that what we see as two very different things—(gain or loss) vs index of refraction—are due to the same physical process.

Now consider the integral along a contour around the upper complex plane:

\[
\int_{-\infty}^{\infty} \frac{\mathcal{E}(w') - \mathcal{E}_0}{w' - z} \, dw'
\]

because this is an analytic expression except for the pole at \( w' = z \), the result of the contour integration is the residue at the pole:

\[
\int_{-\infty}^{\infty} \frac{\mathcal{E}(w') - \mathcal{E}_0}{w' - z} \, dw' = 2\pi i \left[ \mathcal{E}(z) - \mathcal{E}_0 \right]
\]

so we get

\[
\mathcal{E}(z) = \mathcal{E}_0 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\mathcal{E}(w')/\mathcal{E}_0 - 1}{w' - z} \, dw'
\]
Now this expression is valid anywhere in the upper half plane, so we take the limit as the pole approaches the axis where $z = (\omega_1)_{\text{real}}$

The contour then becomes:

This integral has three parts, an integral along the real axis approaching the pole symmetrically, the infinite integral around the bottom of the pole, and the semicircular integral which approaches infinity.

The semicircular integral at infinity goes to zero on physical grounds (Matt cannot respond infinitely fast) (the polarization response to Gamma rays is very small).
The integral along the real axis becomes a principal value integral from $-\infty$ to $+\infty$ and the integral around the infinitesimal contour can be evaluated to $\frac{1}{2}$ the residue of the pole $\zeta$.

\[ \mathcal{E}(\omega) = \mathcal{E}_0 + \frac{1}{\pi i} \text{P.V.} \int_{-\infty}^{\infty} \frac{[\mathcal{E}(\zeta) - \mathcal{E}_0]}{\zeta - \omega} \, d\zeta \]

where $\zeta$ is now constrained to be on the real axis.

Notice that the $i$ in the denominator means that the integral of the real part of $\mathcal{E}$ allows us to find the imaginary part of $\mathcal{E}$, and the integral of the imaginary part of $\mathcal{E}$ allows us to find the real part of $\mathcal{E}$.

\[ \text{Re} \{\mathcal{E}(\omega)\} = \mathcal{E}_0 + \frac{2}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{\text{Re}[\mathcal{E}(\zeta) - \mathcal{E}_0]}{\zeta - \omega} \, d\zeta \]

\[ \text{Im} \{\mathcal{E}(\omega)\} = \frac{2\omega}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{\text{Im}[\mathcal{E}(\zeta) - \mathcal{E}_0]}{\zeta - \omega} \, d\zeta \]
Multiplying the integrals by \( \frac{(z+\omega)}{(z+\omega)} \)

we get

\[
\Re \{ E(\omega) \} = 1 + \frac{1}{\pi c} \text{PV} \int_{-\infty}^{\infty} \frac{E(\omega)}{z^2 - \omega^2} \, dz
\]

\[
\Im \{ E(\omega) \} = -\frac{1}{\pi c} \text{PV} \int_{-\infty}^{\infty} \frac{[\Re \{ E(z) \}] - E_0}{z^2 - \omega^2} \, dz
\]

Now the integrals are expressed in terms of functions which are even \( [\Re \{ E(z) \}] \)

or odd \( [\Im \{ E(z) \}] \), \( z \) \( (z^2 - \omega^2) \),

so we can write the integrals

\[
\Re \{ E(\omega) \} = 1 + \frac{2}{\pi c} \text{PV} \int_{0}^{\infty} \frac{\Im \{ E(z) \}}{z^2 - \omega^2} \, dz
\]

\[
\Im \{ E(\omega) \} = -\frac{2\omega}{\pi c} \text{PV} \int_{0}^{\infty} \frac{\Re \{ E(z) \} - E_0}{z^2 - \omega^2} \, dz
\]
Thus if we have a material which is strongly absorbing over a narrow range of frequencies, we get a dielectric constant which does the following:

\[ \varepsilon \rightarrow \infty \]

In this range, \( \varepsilon \) is approximately independent of frequency, this region is called anomalous dispersion.