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EE 239

Problem Set 3 Solution

4.3 $$m\frac{d^2\xi_e}{dt^2} = -eE_x - m\nu_m\frac{d\xi_e}{dt}$$

use $E_x = \dfrac{en_0\xi_e}{\epsilon_0}$ to get

$$m\frac{d^2\xi_e}{dt^2} = -\frac{e^2n_0}{\epsilon_0}\xi_e - m\nu_m\frac{d\xi_e}{dt}$$

Let $\xi_e(t) = \mathrm{Re}\,\underline{\xi}_0 e^{j\underline{\omega}t}$ { $\underline{\omega}$ can have both real & imaginary parts

$$-\underline{\omega}^2 m\underline{\xi}_0 = -\frac{e^2n_0}{\epsilon_0}\underline{\xi}_0 - j\omega m\nu_m\underline{\xi}_0$$

$$\omega^2 - j\omega\nu_m - \omega_{pe}^2 = 0 \; ; \quad \omega_{pe}^2 = \frac{e^2n_0}{\epsilon_0 m}$$

$$\omega = j\frac{\nu_m}{2} \pm \frac{1}{2}\sqrt{4\omega_{pe}^2 - \nu_m^2} = j\frac{\nu_m}{2} \pm \omega_0$$

For $\nu_m < 2\omega_{pe}$, have a damped oscillation

$$\xi_e(t) = (A\cos\omega_0 t + B\sin\omega_0 t)e^{-\frac{\nu_m}{2}t}$$

$$\frac{d\xi_e}{dt} = \left[\left(\omega_0 B - \frac{\nu_m}{2}A\right)\cos\omega_0 t - \left(\frac{\nu_m}{2}B + \omega_0 A\right)\sin\omega_0 t\right]e^{-\frac{\nu_m}{2}t}$$

At $t=0$, $\dfrac{d\xi_e}{dt} = 0 \Rightarrow B = \dfrac{\nu_m}{2\omega_0}A$

At $t=0$, $\xi_e = \xi_0 \Rightarrow A = \xi_0$

$$\xi_e(t) = \xi_0\left[\cos\omega_0 t + \frac{\nu_m}{2\omega_0}\sin\omega_0 t\right]e^{-\frac{\nu_m}{2}t} \quad //$$

where $\omega_0 = \frac{1}{2}\sqrt{4\omega_{pe}^2 - \nu_m^2}$

4.5

$$\underset{\text{vector}}{\bar{J}_T}(t) = Re(\underset{\text{complex vector}}{\tilde{J}_T} e^{jwt})$$

$$\bar{E}(t) = Re(\tilde{E}\, e^{jwt})$$

$$\tilde{J}_T = \underset{\text{Real part (vector)}}{\bar{J}_{TR}} + j \underset{\text{Imaginary part (vector)}}{\bar{J}_{TI}}$$

$$e^{jwt} = \cos wt + j \sin wt$$

$$\bar{J}_T(t) = \bar{J}_{TR} \cos wt - \bar{J}_{TI} \sin wt \qquad (1)$$

Similarly

$$\bar{E}(t) = \bar{E}_R \cos wt - \bar{E}_I \sin wt \qquad (2)$$

The instantaneous power is (from Maxwell's eqs)

$$P_{abs}(t) = \bar{J}_T(t) \cdot \bar{E}(t) \qquad (3)$$

substitute (1) and (2) into (3) and take the time average $\langle\ \rangle_{wt}$. Note that $\langle \cos^2 wt \rangle = \langle \sin^2 wt \rangle = \frac{1}{2}$; $\langle \sin wt \cos wt \rangle = 0$

$$P_{abs} = \langle P_{abs}(t) \rangle = \frac{1}{2}\left(\bar{J}_{TR} \cdot \bar{E}_R + \bar{J}_{TI} \cdot \bar{E}_I\right)$$

$$= \frac{1}{2} Re\left[(\bar{J}_{TR} + j\bar{J}_{TI}) \cdot (\bar{E}_R - j E_I)\right]$$

$$= \frac{1}{2} Re(\tilde{J}_T \cdot \tilde{E}^*) \checkmark$$

($*$ = complex conjugate)

Similarly

$$P_{abs} = \frac{1}{2} Re\left[(\bar{J}_{TR} - j\bar{J}_{TI}) \cdot (\bar{E}_R + j\bar{E}_I)\right] = \frac{1}{2} Re(\tilde{J}_T^* \cdot \tilde{E}) \checkmark$$

4.6

(a) Assume 1D (no y or z variation)

$$\nabla \times \bar{H} = \bar{J} + \epsilon_0 \frac{\partial \bar{E}}{\partial t} = \bar{J}_T \; ; \quad \bar{J}_T = \hat{x}\, J_0 \cos \omega t$$

$$\nabla \cdot \nabla \times \bar{H} = \nabla \cdot \bar{J}_T = 0 \Rightarrow \frac{d J_0}{dx} = 0 \Rightarrow J_0 = \text{const} \; //$$

(b) $\tilde{J} = J_0 \;, \quad \tilde{E} = E_0 e^{j\phi_0}$

(c) $$\tilde{E}(x) = \frac{\tilde{J}}{\sigma_{dc}(x)} \qquad \sigma_{dc}(x) \approx \frac{e^2 n(x)}{m \nu_m} \quad \text{for } \omega \ll \nu_m$$

$$\tilde{E}(x) = \frac{m \nu_m J_0}{e^2 n_0} \frac{1}{\cos \frac{\pi x}{\ell}} \; //$$

(d) $$P_{ohm}(d) = 2 \int_0^{d/2} \frac{1}{2} J_0 \frac{m \nu_m J_0}{e^2 n_0} dx \sec \frac{\pi x}{\ell}$$

$$P_{ohm}(d) = \frac{m \nu_m}{e^2 n_0} \sigma_0^2 \frac{\ell}{\pi} \ln \underbrace{\left(\sec \frac{\pi d}{2\ell} + \tan \frac{\pi d}{2\ell} \right)}_{\left(\frac{\sin \frac{\pi d}{2\ell} + 1}{\cos \frac{\pi d}{2\ell}} \right)} \; //$$

(e) As $d \to \ell$, $P_{ohm}(d) \to \infty$

$n(x)$ cannot really go to zero at the walls; there is a sheath. //