1. Pole-Zero Doublets

In this problem we will be looking at the behavior of the pseudo-differential amplifier shown below to gain some intuition into the origin and response of pole-zero doublets. You can assume that \( M_1 \) and \( M_2 \) form a differential pair (i.e. their small signal model parameters are identical) with infinite \( r_o \). Moreover, you can assume that \( R_1 \) is larger than \( R_2 \), and that all capacitors are negligible with the exception of the ones explicitly draw in the diagram.

a) Derive an expression for the transfer functions \( H_{11}(s) = \frac{V_{o1}(s)}{V_{i1}(s)} \) and \( H_{22}(s) = \frac{V_{o2}(s)}{V_{i2}(s)} \) in terms of \( R_1, R_2, C \), and the transistors’ \( g_m \).

Note that this amplifier is actually just two separate common-source stages, so we have:

\[
H_{11}(s) = -\frac{g_m R_1}{1 + s R_1 C}, \quad H_{22}(s) = -\frac{g_m R_2}{1 + s R_2 C}
\]

b) Sketch the magnitudes of \( H_{11} \) and \( H_{22} \) versus frequency on the same plot.

Note that in the regime \( s R_1 C \gg 1 \) and \( s R_2 C \gg 1 \), both \( H_{11}(s) \) and \( H_{22}(s) \) reduces to \( g_m/sC \), meaning that they have the same unity-gain frequency. So it’s easy to sketch the bode plot as below:
c) Now derive and sketch the time-domain voltage response of $V_{o1}$ to a voltage step on $V_{i1}$. Similarly, derive and sketch the time-domain voltage response of $V_{o2}$ to a voltage step on $V_{i2}$.

Both $H_{11}$ and $H_{22}$ have only a single pole, so we know that the step response is just a simple exponential settling, so we can write $V_{o1}(t) = A + Be^{-t/\tau}$, where $\tau = R_1C$. Since $V_{o1}(0) = 0$ and $V_{o1}(\infty) = -g_mR_1$, we can conclude that $B = -A = g_mR_1$. We can similarly derive the time waveform for $V_{o2}(t)$. The plots are shown below.
d) Using the results from part a) and part b), sketch the magnitude of the transfer function for differential gain 
\[ H(s) = \frac{(V_{o2} - V_{o1})/(V_{i1} - V_{i2})}{\frac{1}{2} \left( \frac{H_{11}(s)}{1 + sR_1C} + \frac{H_{22}(s)}{1 + sR_2C} \right)} \]

Note that since we want differential gain, we can assume that \( V_{i1} = -V_{i2} = V_d/2 \). Then, substituting and solving we get:

\[
V_{o2} - V_{o1} = -\frac{H_{22}(s)V_d}{2} - \frac{H_{11}(s)V_d}{2} = V_d \left( -\frac{H_{11}(s) + H_{22}(s)}{2} \right)
\]

\[
H(s) = -\frac{H_{11}(s) + H_{22}(s)}{2} = \frac{g_m (R_1 + R_2)}{2} \cdot \frac{1 + \frac{s^{2}R_1R_2}{(R_1 + R_2)C}}{(1 + sR_1C)(1 + sR_2C)}
\]

Note that this transfer function have a pole-zero doublet. However, it is hard to see the plateau behavior for this doublet in the Bode plot (as evident below, where \( R_1 = 10R_2 \)). This is because since \( H(s) \) is the arithmetic average of \( H_{11} \) and \( H_{22} \) (up to a sign change), and \( H_{11} \) is always larger than \( H_{22} \) at every frequency (see part b), the shape of \( H(s) \) is mainly determined by \( H_{11} \) anyways. Mathematically, the separation between the second pole and the zero increase with \( R_1 \), but even with \( R_1 = \infty \), the zero is only half of the second pole frequency, and it is hard to visually see a factor of 2 on log frequency scale.
e) Now sketch the time-domain voltage response of the differential output $V_{o2} - V_{o1}$ to a differential voltage step $V_{i1} - V_{i2}$. While you can certainly derive this response using inverse Laplace transforms and partial fractions, you may find it significantly easier to use your answers from previous sections instead.

Note that in part d, we found that $H(s)$ is the arithmetic average of $H_{11}$ and $H_{22}$ (up to a sign change). Since inverse Laplace transform preserves addition and multiplication by scalar, we conclude:

$$V_o(t) = \frac{g_m R_1}{2} \left(1 - e^{\frac{-t}{R_1 C}}\right) + \frac{g_m R_2}{2} \left(1 - e^{\frac{-t}{R_2 C}}\right)$$

The transient plot is shown below, with each exponential components included too. Note that the settling time is mainly set by the dominant pole.
2. Gain Boosted Cascode

This problem will focus on the gain-boosted cascode amplifier shown below. To simplify the analysis, you can ignore the $r_o$ of the transistors and all of the capacitors except for those explicitly drawn in the diagram.

![Gain Boosted Cascode Amplifier Diagram]

a) What is the frequency response $H(s) = v_3(s)/v_1(s)$ of this amplifier? Approximately what is the unity gain frequency of the amplifier?

First draw the small signal model:

![Small Signal Model Diagram]

The fastest way to figure out the frequency response of this amplifier is just to look at how much of the current from $M_1$ will make it to flow into the output load (i.e., $C_2$). If we look at the input impedance of $M_2$ from node $v_2$, it will be:

$$Z_{i,M2} = \frac{1}{g_{m2} \left(1 - \frac{v_4}{v_2}\right)}, \quad \frac{v_4}{v_2} = -\frac{g_{m3}}{sC_3}$$

$$Z_{i,M2} = \frac{sC_3}{g_{m2}(g_{m3} + sC_3)}$$

Now we can just use the current divider between $Z_{i,M2}$ and $C_1$: 
\[ I_{out,M1} = \frac{1}{sC_1} \cdot I_{M1} \]
\[ I_{out,M1} = \frac{g_{m2}(g_{m3} + sC_3)}{g_{m2}(g_{m3} + sC_3) + s^2C_1C_3} \]
\[ \frac{I_{out,M1}}{I_{M1}} = \frac{1 + \frac{sC_3}{g_{m3}}}{1 + \frac{sC_3}{g_{m3}} + s^2 \left( \frac{C_2C_3}{g_{m2}g_{m3}} \right)} \]

From here, it is easy to see that:

\[ H(s) = \frac{g_{m1}}{sC_2} \cdot \frac{1 + \frac{sC_3}{g_{m3}}}{1 + \frac{sC_3}{g_{m3}} + s^2 \left( \frac{C_2C_3}{g_{m2}g_{m3}} \right)} \]

As long as the amplifier is designed to be stable under unity-gain feedback, the dominant pole of the circuit should be set by the output load, and so the unity gain frequency is still just \( g_{m1}/C_2 \).

b) Approximately what conditions are required to guarantee that the gain boosting feedback loop maintains at least 45º of phase margin? You should provide your answer in terms of \( g_{m1}, g_{m2}, g_{m3}, C_1, C_2, \) and \( C_3 \). What conditions would we obtain if at least 60º of phase margin are required instead?

The unity-gain frequency of the gain-boosting CS amplifier is approximately \( w_u = g_{m3}/C_3 \), and we know that the pole at the source of the cascode transistor is approximately \( w_{p2} = g_{m2}/C_1 \). Therefore, in order to achieve 45 degrees of phase margin, the non-dominant cascode pole must be greater than the unity-gain frequency of the gain-boosting amplifier, i.e.

\[ w_{p2} = \frac{g_{m2}}{C_1} \geq \frac{g_{m3}}{C_3} = w_u \]

In fact, given the loop gain transfer function for the gain-boosting feedback loop:

\[ T(s) = \frac{g_{m3}}{sC_3} \cdot \frac{1}{1 + \frac{sC_1}{g_{m2}}} \]

the phase margin is defined as:

\[ \phi_M = \pi + \angle T(jw_u) = \pi - \frac{\pi}{2} - \arctan \left( \frac{w_u}{w_{p2}} \right) = \frac{\pi}{2} - \arctan \left( \frac{w_u}{w_{p2}} \right) \]

therefore, to achieve a phase margin of 45 degrees, we need:

\[ \frac{\pi}{2} - \arctan \left( \frac{w_u}{w_{p2}} \right) \geq \frac{\pi}{4} \rightarrow w_u \leq w_{p2} \]

If at least 60 degrees of phase margin are required instead, we need:

\[ w_u \leq \frac{w_{p2}}{\sqrt{3}} \rightarrow \frac{g_{m2}}{C_1} \geq \sqrt{3} \cdot \frac{g_{m3}}{C_3} \]
c) Assuming this amplifier is used in unity gain feedback, what conditions are required to guarantee that the gain boosting feedback loop does not introduce any significant pole-zero doublets that might limit the settling response?

What we’re basically trying to ensure is that the gain-boosting loop settles faster than the gain-bandwidth of the overall amplifier. Therefore, we can pretty quickly obtain:

\[
\frac{g_m}{C_3} \geq \frac{g_m}{C_2}
\]

A more mathematical way to arrive at the same solution is realizing that open-loop zeros do not move under feedback. This means that when we close the entire amplifier in unity-gain feedback, if we want to ensure that there is no closed-loop pole-zero doublet limiting the settling, the zero (which is still at \(\frac{g_m}{C_3}\)) must be larger than the closed-loop bandwidth, which is \(\frac{g_m}{C_2}\).

d) Assume now that \(C_1=50\ \text{fF}\), \(C_3=40\ \text{fF}\), \(C_2=1.0\ \text{pF}\), \(V_{m1}^* = 200\ \text{mV}\), and \(V_{m2}^* = 150\ \text{mV}\). In order to achieve at least 60º phase margin and the criteria from part c), what are the minimum and maximum \(\frac{g_m}{g_m}\)?

We have, from part b:

\[
\frac{2I_{M1}}{g_m} = V_{m1}^*, \quad \frac{2I_{M2}}{g_m} = V_{m2}^*, \quad I_{M1} = I_{M2} \rightarrow g_mV_{m1}^* = g_mV_{m2}^* \rightarrow g_m = \frac{V_{m1}^*}{V_{m2}^*}g_m
\]

\[
\frac{g_m}{C_3} \geq \sqrt{3} \cdot \frac{g_m}{C_3} \rightarrow \frac{g_m}{C_3} \leq \frac{C_3}{C_3} \cdot \frac{V_{m1}^*}{V_{m2}^*} \cdot \frac{1}{\sqrt{3}} = 0.6158
\]

And from part c:

\[
\frac{g_m}{C_3} \geq \frac{g_m}{C_2} \rightarrow \frac{g_m}{g_m} \geq \frac{C_3}{C_2} = 0.04
\]

Combining the two:

\[
0.04 \leq \frac{g_m}{g_m} \leq 0.6158
\]

As desired.