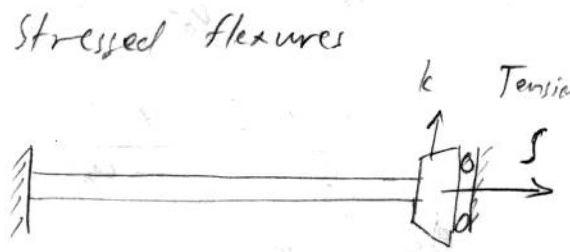
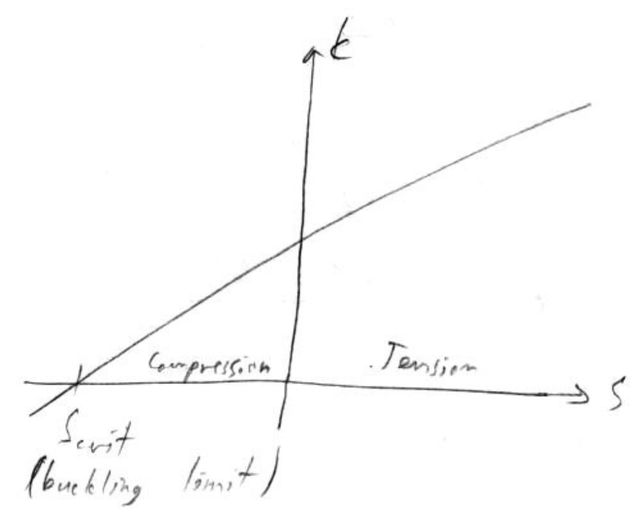
I. Stressed flexures

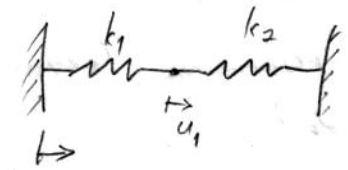


$$k = \frac{pS}{pL - 2\tanh(pL/2)}, \quad p = \sqrt{\frac{S}{EI}}$$



II. Principle of virtual work. Intuitive example.

Find u_1 given u_0.



(Displacement from unstretched position)

First find the extension x in each spring:

$$x_1 = u_1 - u_0, \quad x_2 = -u_1$$

Now find the stored energy:

$$W = \frac{1}{2} k_1 \Delta_1^2 + \frac{1}{2} k_2 \Delta_2^2$$

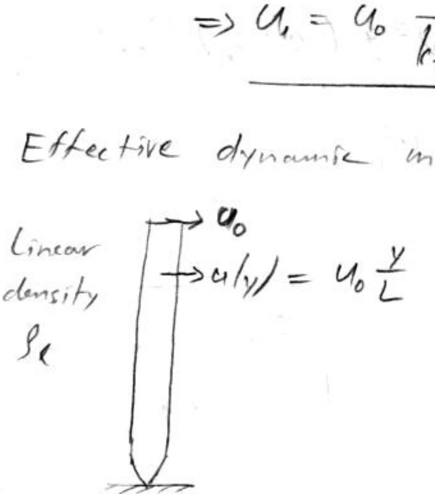
$$= \frac{1}{2}\left(k_1 (u_1 - u_0)^2 + k_2 (u_1)^2\right)$$

Apply principle of virtual work:

$$\frac{\partial W}{\partial u_1} = k_1(u_1 - u_0) + k_2 u_1 \stackrel{!}{=} 0$$

$$\Rightarrow \underline{u_1 = u_0 \frac{k_1}{k_1 + k_2}}$$

III. Effective dynamic mass. Intuitive example.



$$K = \frac{1}{2} m v_0^2$$

$$\Rightarrow m_{eq} = \frac{2K}{v_0^2}$$

Find energy: $K = \frac{1}{2} \int_0^L v^2 \, dm$

$$\Rightarrow K = \frac{1}{2}\int_0^L (\omega_0 u(y))^2 (\rho_\ell \, dy)$$

$$= \frac{\rho_\ell \omega_0^2}{2}\int_0^L \left(u_0 \frac{y}{L}\right)^2 dy$$

$$= \frac{\rho_\ell \omega_0^2 u_0^2}{2L^2} \left(\frac{y^3}{3}\right)\Bigg|_0^L$$

$$K = \frac{\rho_\ell \omega_0^2 u_0^2 L}{6}$$

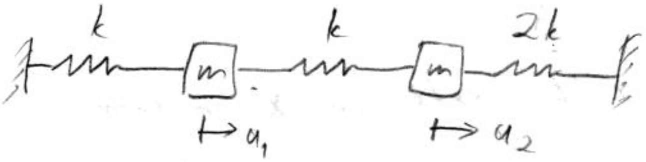
$$m_{eq} = \frac{2K}{v_0^2}$$

$$= \frac{\rho_\ell \omega_0^2 u_0^2 L}{3} \cdot \frac{1}{\omega_0^2 u_0^2}$$

$$\underline{m_{eq} = \frac{\rho_\ell L}{3} = \frac{m}{3}}$$

Implications:

IV. Rayleigh-Ritz Method. Lumped-element example.



Find the resonance frequency. (first mode)

Extensions of springs:

$x_1 = u_1$

$x_2 = u_2 - u_1$

$x_3 = -u_2$

Guess deflections of masses:

$$\hat{u} = \begin{pmatrix} 1 \\ 0.7 \end{pmatrix} u_0$$

$$\Rightarrow u(t) = \begin{pmatrix} 1 \\ 0.7 \end{pmatrix} u_0 e^{j\omega t}$$

$$\Rightarrow v(t) = \begin{pmatrix} 1 \\ 0.7 \end{pmatrix} u_0 \omega j e^{j\omega t}$$

$$K_{max} = \frac{1}{2} m \hat{v}_1^2 + \frac{1}{2} m \hat{v}_2^2$$

$$= \frac{1}{2} m \left((\omega u_0)^2 + (0.7\omega u_0)^2\right)$$

$$= 0.745\, m u_0^2 \omega^2$$

$$W_{max} = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + k x_3^2$$

$$= \frac{1}{2} k\left((u_0)^2 + (-0.3u_0)^2 + 2(-0.7u_0)^2\right)$$

$$= 1.035\, k u_0^2$$

$$0.745\, m u_0^2 \omega^2 = 1.035\, k u_0^2$$

$$\Rightarrow \omega = \sqrt{\frac{1.035\,k}{0.745\,m}}$$

$$= \omega = 1.179\sqrt{\frac{k}{m}}$$

Exact solution:

$$\omega = 1.176\sqrt{\frac{k}{m}}$$