Lecture 15: Beam Combos
Lecture Outline

- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
Stress Gradients in Cantilevers
Vertical Stress Gradients

• Variation of residual stress in the direction of film growth
• Can warp released structures in z-direction
Stress Gradients in Cantilevers

- Below: surface micromachined cantilever deposited at a high temperature then cooled → assume compressive stress

After which, stress is relieved

Once released, beam length increases slightly to relieve average stress → induces moment that bends beam

Stress before release

Stress after release, but before bending

After bending

Stress gradient remains

Average stress
Stress Gradients in Cantilevers (cont)

Find the radius of curvature.

Prior to release, axial stress is: \( \sigma = \sigma_0 - \frac{\sigma_1 z}{(H/2)^2} \) (for the leftmost curve of the previous page)

Find the internal bending moment:

\[
M_x = \int_{-H/2}^{H/2} \left[ (W \cdot d\epsilon) \sigma \right] z 
\]

\[
= W \int_{-H/2}^{H/2} \left( \sigma_0 - \frac{\sigma_1 z}{(H/2)^2} \right) dz
\]

\[
= W \left( \frac{1}{2} \sigma_0 \frac{H^2}{4} - \frac{2 \sigma_1 H^2}{3(8)} \right) - \frac{1}{2} \sigma_0 \frac{H^2}{4} - \frac{1}{6} \sigma_1 WH^2 = M_x
\]
Thus, the radius of curvature is:

\[ \frac{1}{R} = \frac{-M_x}{E'J} \Rightarrow R = \frac{-E'J}{M_x} = -\frac{E'(\frac{1}{12}WH^3)}{6\sigma_iWH^2} = \frac{1}{2} \frac{E'H}{\sigma_i} \]

\[ \text{Biaxial Modulus} \]

\[ [I: \frac{1}{12}WH^3] \]

\[ R = \frac{1}{2} \frac{E'}{(1-\nu)} \frac{H}{\sigma_i} \]

\[ \sigma_i = \frac{1}{2} \frac{E}{(1-\nu)} \frac{H}{R} \]

\[ \Rightarrow R \text{ can be used to determine stress gradient} \]

**Definition.**

Strain gradient, \( \Gamma = \text{slope of the strain-thickness curve} \)

\[ \Gamma = \frac{\sigma_i/E'}{H/2} = \frac{1}{2} \frac{\sigma_i}{H} \frac{1}{E'} \]
Measurement of Stress Gradient

- Use cantilever beams
  - Strain gradient ($\Gamma = \text{slope of stress-thickness curve}$) causes beams to deflect up or down
  - Assuming linear strain gradient $\Gamma$, $z = \Gamma L^2/2$

Where does this come from?
[P. Krulevitch Ph.D.]
To find the amount of bending at the tip of the cantilever:

- go back to:

\[ E I_2 \frac{d^2 w}{dx^2} = -M_x = \frac{1}{6} \sigma_i WH^2 \]

\[ \left[ I_2 = \frac{1}{12} WH^3 \right] \]

\[ \Gamma = \frac{\sigma_i / E'}{H/2} \rightarrow \sigma_i = \Gamma E' \left( \frac{H}{2} \right) \]

\[ \frac{d^2 w}{dx^2} = \Gamma \]

\[ \text{Integrate 2x} \]

\[ \omega = \frac{1}{2} \Gamma x^2 \bigg|_0^L = \frac{1}{2} \Gamma L^2 = \text{Tip Displacement} \]
Folded-Flexure Suspensions
Folded-Beam Suspension

• Use of folded-beam suspension brings many benefits
  ➤ Stress relief: folding truss is free to move in y-direction, so beams can expand and contract more readily to relieve stress
  ➤ High y-axis to x-axis stiffness ratio

Comb-Driven Folded Beam Actuator
# Beam End Conditions

## Table 4.1

<table>
<thead>
<tr>
<th>Type of support</th>
<th>Displacement boundary conditions</th>
<th>Force boundary conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FREE</strong></td>
<td>None</td>
<td>All, as specified</td>
</tr>
<tr>
<td><strong>PINNED</strong></td>
<td>$u = 0$</td>
<td>Moment is specified</td>
</tr>
<tr>
<td></td>
<td>$w = 0$</td>
<td></td>
</tr>
<tr>
<td><strong>ROLLER</strong> (vertical)</td>
<td>$u = 0$</td>
<td>Transverse force and moment are specified</td>
</tr>
<tr>
<td><strong>ROLLER</strong> (horizontal)</td>
<td>$w = 0$</td>
<td>Horizontal force and bending moment are specified</td>
</tr>
<tr>
<td><strong>FIXED or CLAMPED</strong></td>
<td>$u = 0$ $w = 0$ $dw/dx = 0$</td>
<td>None specified</td>
</tr>
</tbody>
</table>

*From Reddy, Finite Element Method*
Common Loading & Boundary Conditions

- Displacement equations derived for various beams with concentrated load $F$ or distributed load $f$


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**cantilever** | **guided-end** | **fixed-fixed**
---|---|---
$x = \frac{F_x L}{E I_w}$ | $x = \frac{F_x L}{E I_w}$ | $x = \frac{F_x L}{4EI_w}$

$y = 4\frac{F_y L^3}{E I_w w^3}$ | $y = \frac{F_y L^3}{E I_w w^3}$ | $y = \frac{1}{16} \frac{F_y L^3}{E I_w w^3}$

$z = 4\frac{F_z L^3}{E I_h h^3}$ | $z = \frac{F_z L^3}{E I_h h^3}$ | $z = \frac{1}{16} \frac{F_z L^3}{E I_h h^3}$

(a) Concentrated load.

---

**cantilever** | **guided-end** | **fixed-fixed**
---|---|---
$x = \frac{f_x L}{E}$ | $x = \frac{f_x L}{E}$ | $x = \frac{f_x L}{4E}$

$y = \frac{3}{2} \frac{f_y L^4}{E I_h w^3}$ | $y = \frac{1}{2} \frac{f_y L^4}{E I_h w^3}$ | $y = \frac{1}{32} \frac{f_y L^4}{E I_h w^3}$

$z = \frac{3}{2} \frac{f_z L^4}{E I_h h^3}$ | $z = \frac{1}{2} \frac{f_z L^4}{E I_h h^3}$ | $z = \frac{1}{32} \frac{f_z L^4}{E I_h h^3}$

(b) Distributed load.
Series Combinations of Springs

- For springs in series with one load
  - Deflections add
  - Spring constants combine like “resistors in parallel”

\[ y(L) = \frac{F}{k} = \frac{y(L_c)}{2} = 2 \left( \frac{F}{k_c} \right) = F \left( \frac{1}{k_c} + \frac{1}{k_c} \right) \]

Stiffness of the Total Beam

Stiffness at the Tip of a Cantilever

Compliances add to give total compliance!
Parallel Combinations of Springs

• For springs in parallel w/ one load
  ▶ Load is shared between the two springs
  ▶ Spring constant is the sum of the individual spring constants

\[ F_a = F/2 \]

\[ k = 2k_a + 2k_b \]

When two springs are in parallel, stiffnesses add!
Folded-Flexure Suspension Variants

- Below: just a subset of the different versions
- All can be analyzed in a similar fashion

[From Michael Judy, Ph.D. Thesis, EECS, UC Berkeley, 1994]
Deflection of Folded Flexures

This equivalent to two cantilevers of length $L_c/2$

Composite cantilever free ends attach here

Half of $F$ absorbed in other half (symmetrical)

4 sets of these pairs, each of which gets $\frac{1}{4}$ of the total force $F$
Constituent Cantilever Spring Constant

- From our previous analysis:

\[ x(y) = \frac{F_c L_c}{2EI_z} y^2 \left(1 - \frac{y}{3L_c}\right) = \frac{F_c y^2}{6EI_z} (3L_c - y) \]

- From which the spring constant is:

\[ k_c = \frac{F_c}{x(L_c)} = \frac{3EI_z}{L^3} \]

- Inserting \( L_c = L/2 \)

\[ k_c = \frac{3EI_z}{(L/2)^3} = \frac{24EI_z}{L^3} \]
Overall Spring Constant

- Four pairs of clamped-guided beams
  - In each pair, beams bend in series
  - (Assume trusses are inflexible)
- Force is shared by each pair → $F_{\text{pair}} = F/4$

**Rigid Truss**

- Force is shared by each pair
- Leg displacement of two legs add
- Thus, springs are in series:

$$\chi = \frac{F_{\text{pair}}}{F_{\text{pair}}} = \frac{1}{k_{\text{leg}} || k_{\text{leg}}} = \left(\frac{F}{4}\right)\left(\frac{1}{k_{\text{leg}}} + \frac{1}{k_{\text{leg}}}\right)$$

- Stiffness of pair
- From before: $k_{\text{leg}} = k_{\text{c}} || k_{\text{c}} = \frac{k_{\text{c}}}{2}$

Thus:

$$\chi = \left(\frac{F}{4}\right)\left(\frac{2}{k_{\text{c}}} + \frac{2}{k_{\text{c}}}\right) = \frac{F}{k_{\text{c}}} = \frac{F}{k_{\text{tot}}}$$

$$k_{\text{tot}} = k_{\text{c}} = \frac{24EI}{L^3}$$