Lecture 16: Energy Methods I
Lecture Outline

• Reading: Senturia, Chpt. 9, 10

• Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
  - Energy Methods
    - Virtual Work
    - Energy Formulations
    - Tapered Beam Example
    - Doubly Clamped Beam Example
    - Large Deflection Analysis
Deflection of Folded Flexures

This equivalent to two cantilevers of length $L_c/2$

Composite cantilever free ends attach here

Half of $F$ absorbed in other half (symmetrical)

4 sets of these pairs, each of which gets $\frac{1}{4}$ of the total force $F$
Constituent Cantilever Spring Constant

• From our previous analysis:

\[
x(y) = \frac{F_c L_c}{2EI_z} y^2 \left(1 - \frac{y}{3L_c}\right) = \frac{F_c y^2}{6EI_z} (3L_c - y)
\]

• From which the spring constant is:

\[
k_c = \frac{F_c}{x(L_c)} = \frac{3EI_z}{L_c^3}
\]

• Inserting \(L_c = L/2\)

\[
k_c = \frac{3EI_z}{(L/2)^3} = \frac{24EI_z}{L^3}
\]
Overall Spring Constant

- Four pairs of clamped-guided beams
  - In each pair, beams bend in series
  - (Assume trusses are inflexible)

- Force is shared by each pair → $F_{\text{pair}} = F/4$

**Formulation:**

Displacement of two legs add thus, springs are in series:

$$\chi = \frac{F_{\text{pair}}}{k_{\text{pair}}} = \frac{F_{\text{pair}}}{(k_{\text{leg}} || k_{\text{leg}})} = \left(\frac{F}{4}\right) \left(\frac{1}{k_{\text{leg}}} + \frac{1}{k_{\text{leg}}}\right)$$

Stiffness of Pair

From before: $k_{\text{leg}} = k_{c} || k_{c} = \frac{k_{c}}{2}$

Thus:

$$\chi = \left(\frac{F}{4}\right) \left(\frac{2}{k_{c}} + \frac{2}{k_{c}}\right) = \frac{F}{k_{c}} = \frac{F}{k_{\text{tot}}}$$

$$\Rightarrow k_{\text{tot}} = k_{c} = \frac{24EIz}{L^3}$$
Folded-Beam Stiffness Ratios

- In the x-direction:
  \[ k_x = \frac{24EI_z}{L^3} \]

- In the z-direction:
  \( k_z = \frac{24EI_x}{L^3} \)  
  \( \Rightarrow \) Same flexure and boundary conditions

- In the y-direction:
  \[ k_y = \frac{8EWh}{L} \]  
  [See Senturia, §9.2]

- Thus:
  \[ \frac{k_y}{k_x} = 4\left(\frac{L}{W}\right)^2 \]  
  Much stiffer in y-direction!
Folded-Beam Suspensions Permeate MEMS

Accelerometer [ADXL-05, Analog Devices]

Gyroscope [Draper Labs.]

Micromechanical Filter [K. Wang, Univ. of Michigan]
Folded-Beam Suspensions Permeate MEMS

• **Below**: Micro-Oven Controlled Folded-Beam Resonator
Stressed Folded-Flexures
Clamped-Guided Beam Under Axial Load

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: \( y(x) \ll L \)

![Beam Diagram]

Governing differential equation: (Euler Beam Equation)

\[
EI_z \frac{d^4 y}{dx^4} - S \frac{d^2 y}{dx^2} = F\delta(x - L)
\]

Axial Load

Unit impulse @ \( x=L \)
Axial stresses produce no net horizontal force; but as soon as the beam is bent, there is a net downward force.

For equilibrium, must postulate some kind of upward load on the beam to counteract the axial stress-derived force.

For ease of analysis, assume the beam is bent to angle $\pi$. 
The Euler Beam Equation

Note: Use of the full bend angle of $\pi$ to establish conditions for load balance; but this returns us to case of small displacements and small angles.
Clamped-Guided Beam Under Axial Load

• Important case for MEMS suspensions, since the thin films comprising them are often under residual stress

• Consider small deflection case: \( y(x) \ll L \)

Governing differential equation: (Euler Beam Equation)

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\]

Axial Load

Unit impulse @ \( x=L \)
Solving the ODE

• Can solve the ODE using standard methods
  ➣ Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)

• Result from Timoshenko:

\[ S > 0 \text{ (tension)} \quad k^{-1} = \frac{pL - 2 \tanh(pL/2)}{p|S|} = \frac{y(x = L)}{F} \]

\[ S < 0 \text{ (compression)} \]

\[ k^{-1} = \frac{-pL + 2 \tan(pL/2)}{p|S|} = \frac{y(x = L)}{F} \]

where \( p = \sqrt{\frac{|S|}{EI_z}} \)
Design Implications

• Straight flexures
  - Large tensile $S$ means flexure behaves like a tensioned wire (for which $k^{-1} = L/S$)
  - Large compressive $S$ can lead to buckling ($k^{-1} \to \infty$)

• Folded flexures
  - Residual stress only partially released
  - Length from truss to shuttle’s centerline differs by $L_S$ for inner and outer legs
Effect on Spring Constant

• Residual compression on outer legs with same magnitude of tension on inner legs:

Beam Strain: \( \varepsilon_b = \pm \varepsilon_r \left( \frac{L_s}{L} \right) \); Stress Force: \( S = \pm E \varepsilon_r \left( \frac{L_s}{L} \right) Wh \)

• Spring constant becomes:

\[
k = 4 \left( k_{com}^{-1} + k_{ten}^{-1} \right)^{-1}
\]

\[
k = 4 \left[ \frac{-pL + 2 \tan \left( \frac{pL}{2} \right)}{p|S|} + \frac{pL - 2 \tanh \left( \frac{pL}{2} \right)}{p|S|} \right]^{-1}
\]

• Remedies:
  - Reduce the shoulder width \( L_s \) to minimize stress in legs
  - Compliance in the truss lowers the axial compression and tension and reduces its effect on the spring constant
Energy Methods
More General Geometries

• Euler-Bernoulli beam theory works well for simple geometries.
• But how can we handle more complicated ones?
• **Example**: tapered cantilever beam
• **Objective**: Find an expression for displacement as a function of location $x$ under a point load $F$ applied at the tip of the free end of a cantilever with tapered width $W(x)$.

$W(x) = W \left(1 - \frac{x}{2L_c}\right)$

50% taper
• In an energy-conserving system (i.e., elastic materials), the energy stored in a body due to the quasi-static (i.e., slow) action of surface and body forces is equal to the work done by these forces ...

• Implication: if we can formulate stored energy as a function of the deformation of a mechanical object, then we can determine how an object responds to a force by determining the shape the object must take in order to minimize the difference $U$ between the stored energy and the work done by the forces:

$$U = \text{Stored Energy} - \text{Work Done}$$

• Key idea: we don’t have to reach $U = 0$ to produce a very useful, approximate analytical result for load-deflection
Fundamentals: Energy Density

• Strain energy density: \([J/m^3]\)
  "To find work done in straining material"

  \[ w = \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x \quad \text{x-axis normal stress term} \]

  \([\sigma_x = E\varepsilon_x]\) \[ w = \int_0^{\varepsilon_x} E\varepsilon_x d\varepsilon_x = \frac{1}{2} E\varepsilon_x^2 \]

• Total strain energy [J]:
  "Integrate over all strains (normal and shear)"

  \[ W = \iiint \left( \frac{1}{2} E(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + \frac{1}{2} G(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2) \right) dV \]
• First, find the bending energy $dW_{bend}$ in an infinitesimal length $dx$:

\[ y(x) = \text{transverse displacement of neutral axis} \]
Energy Due to Axial Load

Strain due to axial load $S$ contributes an energy $dW_{stretch}$ in length $dx$, since lengthening of the different element $dx$ (to $ds$) results in a strain $\varepsilon_x$. 
Shear Strain Energy

\[ W_{\text{shear}} = \frac{3(EL)^2}{4GWh} \int_0^L \left( \frac{d^3y}{dx^3} \right)^2 \, dx \]

Shear Modulus

Applying the Principle of Virtual Work

• **Basic Procedure:**
  - Guess the form of the beam deflection under the applied loads
  - Vary the parameters in the beam deflection function in order to minimize:
    
    $$ U = \sum_j W_j - \sum_i F_i u_i $$

    - Assumes point load
    - Displacement at point load

  - Find minima by simply setting derivatives to zero

• See Senturia, pg. 244, for a general expression with distributed surface loads and body forces
**Example: Tapered Cantilever Beam**

- **Objective**: Find an expression for displacement as a function of location \( x \) under a point load \( F \) applied at the tip of the free end of a cantilever with tapered width \( W(x) \)

\[
W(x) = W \left( 1 - \frac{x}{2L_c} \right)
\]

Top view of cantilever's \( W(x) \)

\[
y(x) = c_2 x^2 + c_3 x^3
\]

- Start by guessing the solution
  - It should satisfy the boundary conditions
  - The strain energy integrals shouldn't be too tedious
    - This might not matter much these days, though, since one could just use matlab or mathematica
Strain Energy And Work By F

\[ U = W_{\text{bend}} - F \cdot y(L_c) \]

\[ W_{\text{bend}} = \frac{1}{2} E \int_{0}^{L_c} I_z(x) \left( \frac{d^2 y}{dx^2} \right)^2 dx \]  

(Bending Energy)

\[ I_z(x) = \frac{W(x)h^3}{12} \]

\[ \frac{d^2 y}{dx^2} = 2c_2 + 6c_3x \]  

(Using our guess)

\[ W(x) = W \left( 1 - \frac{x}{2L_c} \right) \]

Tip Deflection

\[ = \frac{1}{24} EWH^3 \int_{0}^{L_c} \left( 1 - \frac{x}{2L_c} \right) \left( 2c_2 + 6c_3x \right)^2 dx - F \left( c_2L_c^2 + c_3L_c^3 \right) \]
Find $c_2$ and $c_3$ That Minimize $U$

- Minimize $U \rightarrow$ basically, find the $c_2$ and $c_3$ that brings $U$ closest to zero (which is what it would be if we had guessed correctly)

- The $c_2$ and $c_3$ that minimize $U$ are the ones for which the partial derivatives of $U$ with respective to them are zero:

$$\frac{\partial U}{\partial c_2} = 0 \quad \frac{\partial U}{\partial c_3} = 0$$

- Proceed:
  
  First, evaluate the integral to get an expression for $U$:

$$U = EWH^3 \left\{ \frac{5c_3^2}{16} L_c^3 + \frac{c_2 c_3}{3} L_c^2 + \frac{c_2^2}{8} L_c \right\} - F\left(c_2 L_c^2 + c_3 L_c^3 \right)$$
Minimize U (cont)

• Evaluate the derivatives and set to zero:

\[
\frac{\partial U}{\partial c_2} = 0 = \left( \frac{EWh^3}{3} c_3 - F \right) L_c^2 + \left( \frac{EWh^3}{4} c_2 \right) L_c
\]

\[
\frac{\partial U}{\partial c_3} = 0 = \left( \frac{5}{8} EWh^3 c_3 - F \right) L_c^3 + \left( \frac{EWh^3}{3} c_2 \right) L_c^2
\]

• Solve the simultaneous equations to get \( c_2 \) and \( c_3 \):

\[
c_2 = \left( \frac{84}{13} \right) \frac{FL_c}{EWh^3}
\]

\[
c_3 = -\left( \frac{24}{13} \right) \frac{F}{EWh^3}
\]
The Virtual Work-Derived Solution

• And the solution:

\[ y(x) = \left( \frac{24F}{13EWh^3} \right) \left( \frac{7}{2} \right) L_c - x \] \[ x^2 \]

• Solve for tip deflection and obtain the spring constant:

\[ y(L_c) = \left( \frac{24F}{13EWh^3} \right) \left( \frac{5}{2} \right) L_c^3 \]

\[ k_c = F / y(L_c) = \left( \frac{13EWh^3}{60L_c^3} \right) \]

• Compare with previous solution for constant-width cantilever beam (using Euler theory):

\[ y(L_c) = \left( \frac{4F}{EWh^3} \right) L_c^3 \rightarrow 13\% \text{ smaller than tapered-width case} \]
Comparison With Finite Element Simulation

• Below: ANSYS finite element model with
  \[ L = 500 \, \mu m \quad W_{\text{base}} = 20 \, \mu m \quad E = 170 \, \text{GPa} \]
  \[ h = 2 \, \mu m \quad W_{\text{tip}} = 10 \, \mu m \]

• **Result:** (from static analysis)
  \[ k = 0.471 \, \mu N/m \]

• This matches the result from energy minimization to 3 significant figures
Need a Better Approximation?

• Add more terms to the polynomial
• Add other strain energy terms:
  - Shear: more significant as the beam gets shorter
  - Axial: more significant as deflections become larger
• Both of the above remedies make the math more complex, so encourage the use of math software, such as Mathematica, Matlab, or Maple
• Finite element analysis is really just energy minimization
• If this is the case, then why ever use energy minimization analytically (i.e., by hand)?
  - Analytical expressions, even approximate ones, give insight into parameter dependencies that FEA cannot
  - Can compare the importance of different terms
  - Should use in tandem with FEA for design