Lecture 18: Resonance Frequency
Lecture Outline

• Reading: Senturia, Chpt. 10
• Lecture Topics:
  - Energy Methods
    - Virtual Work
    - Energy Formulations
    - Tapered Beam Example
    - Estimating Resonance Frequency
Estimating Resonance Frequency
Clamped-Clamped Beam \( \mu \)Resonator

Resonator Beam

Electrode

Sinusoidal Excitation

Voltage-to-Force Capacitive Transducer

\[ \nu_i = V_i \cos[\omega_o t] \rightarrow f_i = F_i \cos[\omega_o t] \]

- \( \omega \neq \omega_o \): small amplitude
- \( \omega = \omega_o \): maximum amplitude \( \rightarrow \) beam reaches its maximum potential and kinetic energies

\[ V_P \]

\[ i_o \]

\[ i_o \]

\[ Q \approx 10,000 \]
Estimating Resonance Frequency

- Assume simple harmonic motion:

\[ x(t) = x_0 \cos(\omega t) \]

- Potential Energy:

\[ W(t) = \frac{1}{2} k x^2(t) = \frac{1}{2} k x_0^2 \cos^2(\omega t) \]

- Kinetic Energy:

\[ K(t) = \frac{1}{2} M x^2(t) = \frac{1}{2} M x_0^2 \omega^2 \sin^2(\omega t) \]
Estimating Resonance Frequency (cont)

• Energy must be conserved:
  - Potential Energy + Kinetic Energy = Total Energy
  - Must be true at every point on the mechanical structure

  Occurs at peak displacement

  \[ W_{\text{max}} = \frac{1}{2} kx_o^2 \]

  Maximum Potential Energy

  Stiffness

  Displacement Amplitude

  Occurs when the beam moves through zero displacement

  \[ K_{\text{max}} = \frac{1}{2} M\omega^2 x_o^2 \]

  Maximum Kinetic Energy

  Mass

  Radian Frequency

• Solving, we obtain for resonance frequency:

  \[ \omega = \sqrt{\frac{k}{M}} \]
Example: ADXL-50

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: $L = 260 \, \mu m$, $h = 2.3 \, \mu m$, $W = 2 \, \mu m$
Lumped Spring-Mass Approximation

- Mass is dominated by the proof mass
  - 60% of mass from sense fingers
  - Mass = \( M = 162 \text{ ng} \) (nano-grams)

- Suspension: four tensioned beams
  - Include both bending and stretching terms [A.P. Pisano, BSAC Inertial Sensor Short Courses, 1995-1998]

\[ k_c \rightarrow \text{stiffness of a cantilever} \]
\[ F/4 \]

\[ \text{Bending compliance} \ k_b^{-1} \]

\[ \text{Stretching compliance} \ k_{st}^{-1} \]
ADXL-50 Suspension Model

- **Bending contribution:**
  \[ k_b^{-1} = \left( \frac{1}{k_c} + \frac{1}{k_o} \right) = \frac{L^3}{EWh^3} = 4.2 \mu m/\mu N \]

- **Stretching contribution:**
  \[ k_{st}^{-1} = \frac{L}{S} = \frac{L}{6Wh} = 1.14 \mu m/\mu N \]

- **Total spring constant:** add bending to stretching (stiffnesses are in parallel) + (4 beams in parallel)
  \[ k = 4(k_b + k_{st}) = 4(0.24 + 0.8) = 4.48 \mu N/\mu m \]

\[ F_y = S \sin \theta = S(x/L) = \left( \frac{S}{L} \right)x \]
• Using a lumped mass-spring approximation:

\[ f = \frac{1}{2\pi} \sqrt{\frac{k}{M}} = \frac{1}{2\pi} \sqrt{\frac{4.48N/m}{162 \times 10^{-12} \text{ kg}}} = 26.5kHz \]

• On the ADXL-50 Data Sheet: \( f_o = 24 \text{ kHz} \)

\( \Rightarrow \) Why the 10% difference?

\( \Rightarrow \) Well, it’s approximate … plus …

\( \Rightarrow \) Above analysis does not include the frequency-pulling effect of the DC bias voltage across the plate sense fingers and stationary sense fingers … something we’ll cover later on …
• Vibrating structure displacement function:

\[ y(x, t) = \hat{y}(x) \cos(\omega t) \]

Maximum displacement function (i.e., mode shape function)
Seen when velocity \( \dot{y}(x, t) = 0 \)

• Procedure for determining resonance frequency:
  ➤ Use the static displacement of the structure as a trial function and find the strain energy \( W_{\text{max}} \) at the point of maximum displacement (e.g., when \( t=0, \pi/\omega, ... \))
  ➤ Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  ➤ Equate energies and solve for frequency
Maximum Kinetic Energy

- **Displacement:** \( y(x, t) = \hat{y}(x) \cos[\omega t] \)

- **Velocity:** \( v(x, t) = \frac{\partial y(x, t)}{\partial t} = -\omega \hat{y}(x) \sin[\omega t] \)

- At times \( t = \pi/(2\omega), 3\pi/(2\omega), \ldots \)

\[ \hat{y}(x, t) = 0 \]

The displacement of the structure is \( y(x, t) = 0 \)

The velocity is maximum and all of the energy in the structure is kinetic (since \( \mathcal{W} = 0 \)):

\[ v(x, \pi/(\hbar \omega)) = -\omega \hat{y}(x) \]
Maximum Kinetic Energy (cont)

• At times $t = \pi/(2\omega)$, $3\pi/(2\omega)$, ...

Velocity: $v(x, \frac{n\pi}{(n\omega)}) = -\omega \hat{y}(x)$

$\frac{dK}{dt} = \frac{1}{2} \cdot dm \cdot [v(x,t)]^2$

$dm = \rho (Wh \cdot dx)$

$= Maximum Kinetic Energy:$

$K_{max} = \int_0^L \frac{1}{2} \rho Wh dx \cdot v^2(x, t) = \int_0^L \frac{1}{2} \rho Wh \omega^2 \hat{y}(x)^2 dx$

(time when velocity is maximum)
The Raleigh-Ritz Method

• Equate the maximum potential and maximum kinetic energies:

\[ K_{\text{max}} = \int_0^L \frac{1}{2} \rho Wh \omega^2 \hat{y}^2(x) \, dx = W_{\text{max}} \]

• Rearranging yields for resonance frequency:

\[ \omega = \sqrt{\frac{L}{\int_0^L \frac{1}{2} \rho Wh \hat{y}^2(x) \, dx}} \]

\( \omega = \) resonance frequency
\( W_{\text{max}} = \) maximum potential energy
\( \rho = \) density of the structural material
\( W = \) beam width
\( h = \) beam thickness
\( \hat{y}(x) = \) resonance mode shape
Example: Folded-Beam Resonator

- Derive an expression for the resonance frequency of the folded-beam structure at left.

Use Rayleigh-Ritz method:

\[ \text{KE}_{\text{max}} = \text{PE}_{\text{max}} \]

**Kinetic Energy:**

\[ \text{KE}_{\text{max}} = \text{KE}_s + \text{KE}_t + \text{KE}_b \]

- shuttle
- trusses
- beams

\[ = \frac{1}{2} \omega_s^2 M_s + \frac{1}{2} \omega_t^2 M_t + \frac{1}{2} \int \omega_b^2 dM_b \]

**Anchor**

- mass \( M_t/2 \)

**Shuttle**

- mass \( M_s \)

**Folded truss**

- mass \( M_b \)

- anchor height \( h = \text{thickness} \)

- maximum displacement of the shuttle \( x_0 \)
Get Kinetic Energies

Velocity of the shuttle: \( V_s = \omega_0 x_0 \)

Velocity of the truss: \( V_t = \frac{1}{2} N_s^2 M_s + \frac{1}{2} \omega_0^2 x_0^2 M_s \)

Velocity of the beam segments:

\( V_t = \frac{1}{2} (\frac{1}{2} \omega_0 x_0)^2 M_t + \frac{1}{8} \omega_0^2 x_0^2 M_t \)

Assume the mode shape is the same as the static displacement shape:

For segment AB

\( \hat{x}(y) = \frac{F_x}{48EI_2} (3Ly^2 - 2y^3) \quad 0 \leq y \leq L \quad 0 \)
Folded-Beam Suspension

Comb-Driven Folded Beam Actuator

\[ \chi(y) = \frac{F_x}{48EI} \left(3Ly^2 - 2y^3 \right) \quad 0 \leq y \leq L \]

Case: \( y = 0 \) \( \chi(y=0) = 0 \)

Case: \( y \cdot L \leq L \) \( \chi(y \cdot L) = \frac{F_x}{48EI} \cdot \frac{L^3}{4L} \cdot \frac{L^2}{L^3} = 2k_{IC} \)
Get Kinetic Energies (cont)

At \( y = L \): \( \hat{x}(L) = \frac{X_0}{2} = \frac{F_x L^3}{4 \pi E I_z} \)

Substituting into (1):

\[
\hat{x}(y) = \frac{X_0}{2} \left[ 3 \left(\frac{L}{y}\right)^2 - 2 \left(\frac{L}{y}\right)^3 \right]
\]

Which yields for velocity:

\[
\sqrt{b}(y) \mid_{[AB]} = \frac{X_0}{2} \left[ 3 \left(\frac{L}{y}\right)^2 - 2 \left(\frac{L}{y}\right)^3 \right] \omega_0
\]

Plugging into the expression for \( KE_{b} \):

\[
KE_{[AB]} = \frac{1}{2} \int_0^L \frac{X_0^2 \omega_0^2}{4} \left[ 3 \left(\frac{L}{y}\right)^2 - 2 \left(\frac{L}{y}\right)^3 \right]^2 dM_{[AB]}
\]

[uniform material] \( \Rightarrow dM_{[AB]} = \frac{M_{[AB]}}{L} dy \)

\[
v = \frac{X_0^2 \omega_0^2 M_{[AB]}}{8L} \int_0^L \left[ 3 \left(\frac{L}{y}\right)^2 - 2 \left(\frac{L}{y}\right)^3 \right]^2 dy
\]

\[
KE_{[AB]} = \frac{13}{280} \frac{X_0^2 \omega_0^2 M_{[AB]}}{L^2}
\]
Get Kinetic Energies (cont)

For segment CD:

\[ u_b(y) \bigg|_{CD} = X_0 \left[ 1 - \frac{3}{2} \left( \frac{y}{L} \right)^2 + \left( \frac{y}{L} \right)^3 \right] \omega_0 \]

Thus:

\[ KE_{[CD]} = \frac{X_0^2 \omega_0^2 M_{[CD]}}{2L} \int_0^L \left[ 1 - \frac{3}{2} \left( \frac{y}{L} \right)^2 + \left( \frac{y}{L} \right)^3 \right] dy \]

\[ KE_{[CD]} = \frac{83}{280} X_0^2 \omega_0^2 M_{[CD]} \]

Let \( M_b \) = total mass of the 8 beams.

Then:

\[ M_{[AB]} = M_{[CD]} = \frac{1}{8} M_b \]

Thus:

\[ KE_6 = 4 KE_{[AB]} + 4 KE_{[CD]} = \frac{6}{35} X_0^2 \omega_0^2 M_b \]

and

\[ KE_{max} = X_0^2 \omega_0^2 \left[ \frac{1}{2} M_s + \frac{1}{6} M_t + \frac{6}{35} M_b \right] \]

Folded-beam suspension

Shuttle w/ mass \( M_s \)

Folding truss w/ mass \( M_t \) \( \frac{1}{2} \)

Anchor \( h = \) thickness
Get Potential Energy & Frequency

Folded-beam suspension

Shuttle w/ mass $M_s$

Folding truss w/ mass $M_t/2$

Anchor

$h = \text{thickness}$

PE$_{\text{max}}$ is simply the work done to achieve maximum deflection:

$$PE_{\text{max}} = \frac{1}{2} k_x x_0^2$$

Thus, using Raleigh-Ritz:

$$KE_{\text{max}} = PE_{\text{max}}$$

$$x_0^2 \omega_0^2 \left[ \frac{1}{2} M_s + \frac{1}{8} M_t + \frac{6}{35} M_b \right] = \frac{1}{2} k_x x_0$$

$$\omega_0 = \left[ \frac{k_x}{M_{\text{eq}}} \right]^{\frac{1}{2}}$$

Where $M_{\text{eq}} = M_s + \frac{1}{8} M_t + \frac{12}{35} M_b$

(Resonance Frequency of a Folded-Beam Suspended Shuttle)