Lecture 19: Resonance Frequency II
Lecture Outline

• Reading: Senturia, Chpt. 10
• Lecture Topics:
  ◦ Energy Methods
    ◦ Virtual Work
    ◦ Energy Formulations
    ◦ Tapered Beam Example
    ◦ Estimating Resonance Frequency
The Raleigh-Ritz Method

• Equate the maximum potential and maximum kinetic energies:

\[ K_{\text{max}} = \frac{1}{2} \rho W h \omega^2 \hat{y}^2(x) dx = W_{\text{max}} \]

• Rearranging yields for resonance frequency:

\[ \omega = \sqrt{\frac{W_{\text{max}}}{\int_0^L \frac{1}{2} \rho W h \hat{y}^2(x) dx}} \]

\( \omega = \) resonance frequency
\( W_{\text{max}} = \) maximum potential energy
\( \rho = \) density of the structural material
\( W = \) beam width
\( h = \) beam thickness
\( \hat{y}(x) = \) resonance mode shape
Example: Folded-Beam Resonator

• Derive an expression for the resonance frequency of the folded-beam structure at left.

Use Rayleigh–Ritz method:

\[ KE_{\text{max}} = PE_{\text{max}} \]

Kinetic Energy:

\[ KE_{\text{max}} = KE_s + KE_t + KE_b \]

- Shuttle: \( KE_s = \frac{1}{2} M_s \dot{x}_0^2 \)
- Trusses: \( KE_t = \frac{1}{2} M_t \dot{x}_0^2 \)
- Beams: \( KE_b = \frac{1}{2} \int M_b \dot{x}_0^2 \, dx \)

Max displacement of the shuttle: \( x_0 \)

Mass of both trusses: \( M_t \)

Must integrate since the beam velocity is a function of location \( y \)!
Get Kinetic Energies

Velocity of the shuttle: \( V_s = \omega_0 X_0 \)

Resonance Freq. \( \omega_0 \)

Maximum Displacement Amplitude

\[ KE_s = \frac{1}{2} N_s^2 M_s = \frac{1}{2} \omega_0^2 X_0^2 M_s \]

Velocity of the truss: \( V_t = \frac{1}{2} N_s \cdot \frac{1}{2} \omega_0 X_0 \)

\[ KE_t = \frac{1}{2} \left( \frac{1}{2} \omega_0 X_0 \right)^2 M_t = \frac{1}{8} \omega_0^2 X_0^2 M_t \]

Velocity of the beam segments:

\[ X(y) = \frac{F_x}{3EI} (3Ly^2 - 2y^3) \quad 0 \leq y \leq L \quad (0) \]

Assume the mode shape is the same as the static displacement shape.

\[ \text{For segment AB} \]

Anchor \( h = \text{thickness} \)

Folded-beam suspension

Shuttle w/ mass \( M_s \)

Folding truss w/ mass \( M_t \)

Anchor
Folded-Beam Suspension

Comb-Driven Folded Beam Actuator

\[ \chi(y) = \frac{F_x}{48EI_z} \left( 3L^2 - 2y^3 \right) \quad 0 \leq y \leq L \]

Case: \( y = 0 \) \( \chi(y=0) = 0 \)

Case: \( y = L \) \( \chi(y=L) = \frac{F_x}{48EI_z} L^3 \rightarrow k = \frac{F_x L^4}{12EI_z L} = \frac{12EI_L}{L^3} = \frac{k_c}{2} \)
Get Kinetic Energies (cont)

At $y = L$: \( \hat{x}(L) = \frac{X_0}{2} = \frac{F x L^3}{48EI_2} \)

Substituting into (1):

\[
\hat{x}(y) = \frac{X_0}{2} \left[ 3 \left( \frac{y}{L} \right)^2 - 2 \left( \frac{y}{L} \right)^3 \right]
\]

which yields for velocity:

\[
\sqrt{b}(y) \bigg|_{[AB]} = \frac{X_0}{2} \left[ 3 \left( \frac{y}{L} \right)^2 - 2 \left( \frac{y}{L} \right)^3 \right] \omega_0
\]

Plugging into the expression for $KE_b$:

\[
KE_{[AB]} = \frac{1}{2} \int_0^L \frac{X_0 \omega_0^2}{4} \left[ 3 \left( \frac{y}{L} \right)^2 - 2 \left( \frac{y}{L} \right)^3 \right]^2 \, dm_{[AB]}
\]

For $[AB] = \text{static}$ mass of beam:

\[
F \text{olding truss w/ mass } M_t\frac{1}{2}
\]

\[
KE_{[AB]} = \frac{13}{280} X_0 \omega_0^2 M_{[AB]}
\]
Get Kinetic Energies (cont)

For segment CD:

\[ V_b(y) \big|_{(CD)} = X_0 \left[ 1 - \frac{3}{2} \left( \frac{y}{L} \right)^2 + \left( \frac{y}{L} \right)^3 \right] \omega_0 \]

Thus:

\[ K_E_{(CD)} = \frac{X_0^2 \omega_0^2 M_{[CD]}}{2L} \int_0^L \left[ 1 - \frac{3}{2} \left( \frac{y}{L} \right)^2 + \left( \frac{y}{L} \right)^3 \right] dy \]

\[ K_E_{(CD)} = \frac{83}{280} X_0^2 \omega_0^2 M_{[CD]} \]

Let \( M_b \) denote total mass of the \( \delta \) beams.

Then:

\[ M_{[AB]} = M_{(CD)} = \frac{1}{8} M_b \]

Thus:

\[ K_E_6 = 4 K_E_{[AB]} + 4 K_E_{[CD]} = \frac{6}{35} X_0^2 \omega_0^2 M_b \]

and

\[ K_{E_{\text{max}}} = X_0^2 \omega_0^2 \left[ \frac{1}{2} M_s + \frac{1}{8} M_t + \frac{6}{35} M_b \right] \]
Get Potential Energy & Frequency

PE_{max} is simply the work done to achieve maximum deflection: $PE_{max} = \frac{1}{2} k_x X_0^2$

Thus, using Raleigh-Ritz:

$X_0^2 \omega_0^2 \left[ \frac{1}{2} M_s + \frac{1}{8} M_t + \frac{6}{35} M_b \right] = \frac{1}{2} k_x X_0$

$\omega_0 = \left[ \frac{k_x}{M_{eq}} \right]^{\frac{1}{2}}$

where $M_{eq} = M_s + \frac{1}{4} M_t + \frac{12}{35} M_b$

(Resonance Frequency of a Folded-Beam Suspended Shuttle)
Brute Force Methods for Resonance Frequency Determination
Basic Concept: Scaling Guitar Strings

Guitar String

- Vibrating “A” String (110 Hz)
- Freq. Equation:
  \[ f_0 = \frac{1}{2\pi} \sqrt{\frac{k_r}{m_r}} \]

μMechanical Resonator

- Metallized Electrode
- Anchor
- Polysilicon Clamped-Clamped Beam
- Performance:
  - \( k_r = 8.5 \text{ MHz} \)
  - \( Q_{vac} = 8,000 \)
  - \( Q_{air} \approx 50 \)
  - \( L_r = 40.8 \mu\text{m} \)
  - \( m_r \approx 10^{-13} \text{ kg} \)
  - \( W_r = 8 \mu\text{m}, h_r = 2 \mu\text{m} \)
  - \( d = 1000 \text{ Å}, V_P = 5 \text{ V} \)
  - \( \text{Press.} = 70 \text{ mTorr} \)

[Bannon 1996]
Anchor Losses

**Problem:**
- Fixed-Fixed Beam Resonator: direct anchoring to the substrate → anchor radiation into the substrate → lower Q

**Solution:**
- Support at motionless nodal points → isolate resonator from anchors → less energy loss → higher Q

**Fixed-Fixed Beam Resonator**
- $Q = 300$ at 70MHz

**Free-Free Beam Resonator**
- $Q = 15,000$ at 92MHz

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EE C245: Introduction to MEMS Design  Lecture 19  C. Nguyen  11/4/08
92 MHz Free-Free Beam \( \mu\)Resonator

- Free-free beam \( \mu\)mechanical resonator with non-intrusive supports ⇨ reduce anchor dissipation ⇨ higher \( Q \)

**Design/Performance:**
- \( L_r = 13.1 \mu m, W_r = 6 \mu m \)
- \( h = 2 \mu m, d = 1000 \AA \)
- \( V_p = 28-76V, W_o = 2.8 \mu m \)
- \( f_0 \sim 92.25 MHz \)
- \( Q \sim 7,450 @ 10m\)Torr

[Wang, Yu, Nguyen 1998]
Higher Order Modes for Higher Freq.

2\textsuperscript{nd} Mode Free-Free Beam

3\textsuperscript{rd} Mode Free-Free Beam

Distinct Mode Shapes

Electrodes

Ancho

Support Beam

\( L_r = 20.3 \ \mu m \)

Ancho

\( h = 2.1 \ \mu m \)

\( L_r = 20.3 \ \mu m \)

\( h = 2.1 \ \mu m \)

\( Q = 11,500 \)

Transmission [dB]

Frequency [MHz]

Phase [degree]
**Flexural-Mode Beam Wave Equation**

- **Transverse Displacement**: $u$
- **Width**: $W$

\[ \rho A \frac{\partial^2 u}{\partial t^2} = ma \]

**Derive the wave equation for transverse vibration**

**Dynamic Equilibrium Condition for Forces in the $x$-direction**:

\[ F - (F + \frac{\partial F}{\partial x} \, dx) - \rho A dx \frac{\partial^2 u}{\partial t^2} = 0 \quad (1) \]

**Do the same for moments**:

\[ -Fdx + \frac{\partial M}{\partial x} \, dx = 0 \quad (2) \]

Combining (1) and (2):

\[ \frac{\partial^4 u}{\partial x^4} = -\rho A \frac{\partial^2 u}{\partial t^2} \]

\[ \frac{\partial^2 u}{\partial x^2} = -\frac{M}{EI} \rightarrow M = -EI \frac{\partial^2 u}{\partial x^2} \]
Example: Free-Free Beam

- Determine the resonance frequency of the beam
- Specify the lumped parameter mechanical equivalent circuit
- Transform to a lumped parameter electrical equivalent circuit
- Start with the flexural-mode beam equation:

\[
\frac{\partial^2 u}{\partial t^2} = \left( \frac{EI}{\rho A} \right) \frac{\partial^4 u}{\partial x^4}
\]
Free-Free Beam Frequency

• Substitute \( u = u_1 e^{j\omega t} \) into the wave equation:

\[
\frac{\partial^4 u}{\partial x^4} = \left( \frac{\omega^2 \rho A}{EI} \right) u \tag{1}
\]

• This is a 4\(^{th}\) order differential equation with solution:

\[
u(x) = A \cosh kx + B \sinh kx + C \cos kx + D \sin kx \tag{2}
\]

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**Gives the mode shape during resonance vibration.**

• Boundary Conditions:

\[
\begin{align*}
\text{At } x &= 0 & \text{At } x &= \ell \\
\frac{\partial^2 u}{\partial x^2} &= 0 & \frac{\partial^2 u}{\partial x^2} &= 0 \\
\frac{\partial^3 u}{\partial x^3} &= 0 & \frac{\partial^3 u}{\partial x^3} &= 0 & M &= 0 \text{ (Bending moment)} \\
\frac{\partial M}{\partial x} &= 0 & \text{(Shearing force)}
\end{align*}
\]
• Applying B.C.’s, get $A=C$ and $B=D$, and
\[
\begin{pmatrix}
(cosh \, k\ell - cos \, k\ell) & (sinh \, k\ell - sin \, k\ell) \\
(sinh \, k\ell + sin \, k\ell) & (cosh \, k\ell - cos \, k\ell)
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix} = 0
\]  
(3)

• Setting the determinant $= 0$ yields
\[
\cos k\ell = \frac{1}{cosh \, k\ell}
\]

• Which has roots at
\[k_1\ell = 4.730 \quad k_2\ell = 7.853 \quad k_3\ell = 10.996\]

• Substituting (2) into (1) finally yields:
\[k^4 = \frac{\rho A}{EI} \omega^2 \longrightarrow f_n = \frac{(k_n\ell)^2}{2\pi^2\ell^2} \sqrt{\frac{EI}{\rho A}}\]
## Higher Order Free-Free Beam Modes

<table>
<thead>
<tr>
<th>Mode</th>
<th>( n )</th>
<th>Nodal Points</th>
<th>( k_n \ell )</th>
<th>( f_n/f_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fundamental ( (f_1) )</td>
<td>1</td>
<td>2</td>
<td>4.730</td>
<td>1.000</td>
</tr>
<tr>
<td>1st Harmonic</td>
<td>2</td>
<td>3</td>
<td>7.853</td>
<td>2.757</td>
</tr>
<tr>
<td>2nd Harmonic</td>
<td>3</td>
<td>4</td>
<td>10.996</td>
<td>5.404</td>
</tr>
<tr>
<td>3rd Harmonic</td>
<td>4</td>
<td>5</td>
<td>14.137</td>
<td>8.932</td>
</tr>
<tr>
<td>4th Harmonic</td>
<td>5</td>
<td>6</td>
<td>17.279</td>
<td>13.344</td>
</tr>
</tbody>
</table>

More than 10x increase

**Fundamental Mode (n=1)**

**1\textsuperscript{st} Harmonic (n=2)**

**2\textsuperscript{nd} Harmonic (n=3)**
• The mode shape expression can be obtained by using the fact that $A=C$ and $B=D$ into (2), yielding

$u_x = \mathcal{D} \left[ \left( \frac{A}{D} \right) (\cosh kx + \cos kx) + (\sinh kx + \sin kx) \right]$

• Get the amplitude ratio by expanding (3) [the matrix] and solving, which yields

$\frac{A}{D} = \frac{\sin kl - \sinh kl}{\cosh kl - \cos kl}$

• Then just substitute the roots for each mode to get the expression for mode shape

Fundamental Mode ($n=1$)

[Substitute $k_1l = 4.730$ ]
Lumped Parameter Mechanical Equivalent Circuit
Equivalent Dynamic Mass

• Once the mode shape is known, the lumped parameter equivalent circuit can then be specified
• Determine the equivalent mass at a specific location $x$ using knowledge of kinetic energy and velocity

Equivalent Mass = $M_{eqx} = \frac{K.E.}{\frac{1}{2}V_x^2} = \frac{\frac{1}{2} \rho A \int_0^L V_x^2(x) \, dx}{\frac{1}{2} V_x^2}$

Maximum Kinetic Energy

Maximum Velocity @ location $x$