EE C245 – ME C218
Introduction to MEMS Design
Fall 2007

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Lecture 22: Capacitive Transducers

Logistics

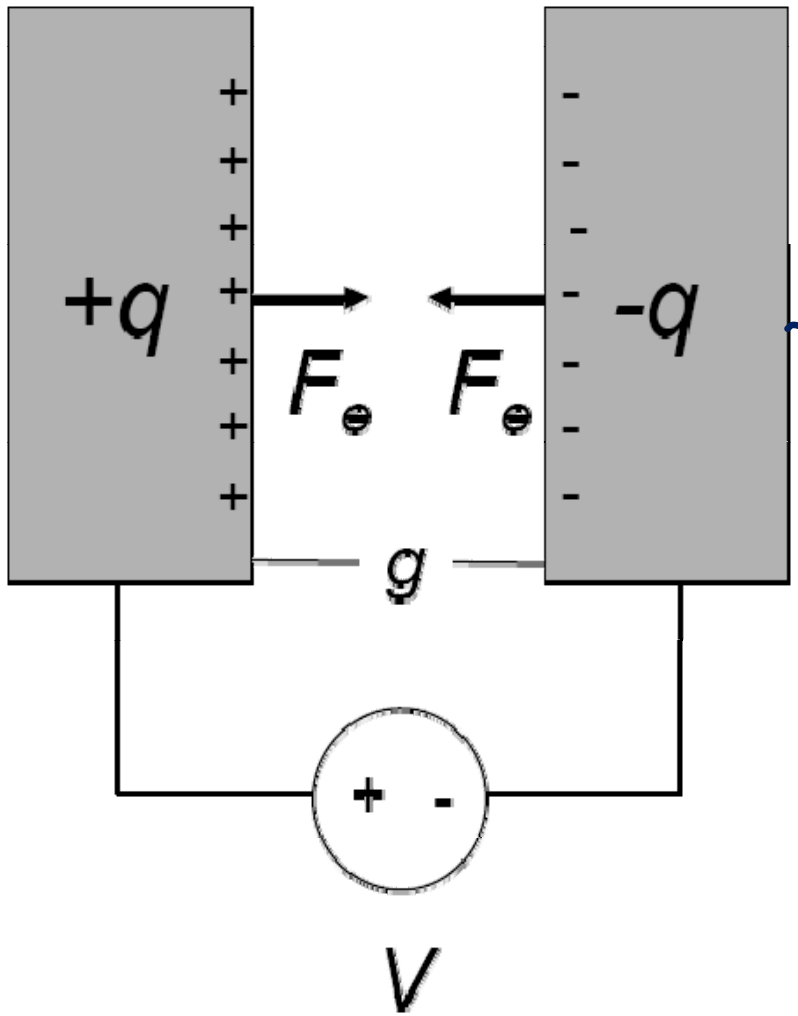
- All EE's to one side; ME's to the other
- Email me your group makeups
- Makeup/Extra Lecture: During next Monday's discussion

Lecture Outline

- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis

Energy Conserving Transducers

Basic Physics of Electrostatic Actuation



- <u>Goal</u>: Determine gap spacing g as a function of input variables
- First, need to determine the energy of the system
- Two ways to change the energy:
  - Change the charge q
  - Change the separation g

$$\Delta W(q,g) = V\Delta q + F_e\Delta g$$

$$dW = Vdq + F_e dg$$

- <u>Note</u>: We assume that the plates are supported elastically, so they don't collapse

Stored Energy

- Here, the stored energy is the work done in increasing the gap after charging capacitor at zero gap

$$\mathcal{W} = 0 + \int_0^g F_e dg'$$

No change in charge: $dq = 0$

$\mathcal{E} = \frac{q}{\epsilon A}$

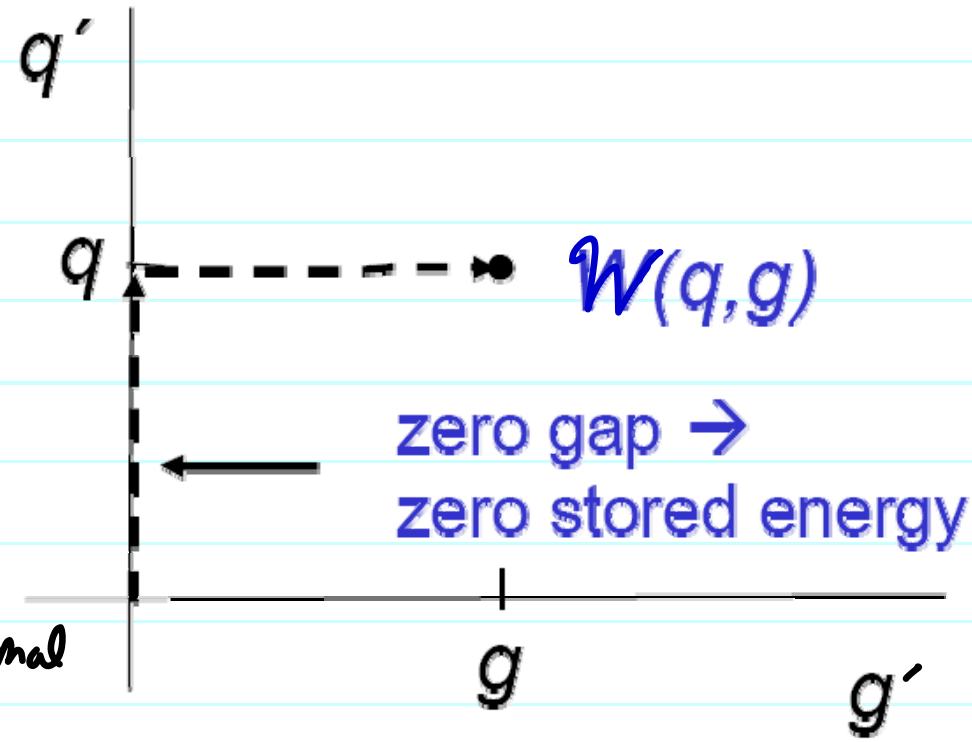
Cross-sectional Area

$F_e = \left(\frac{q}{2}\right)\mathcal{E} = \frac{1}{2}\frac{q^2}{\epsilon A}$

(independent of g)

$\therefore \mathcal{W} = \int_0^g F_e dg' = F_e g'\Big|_0^g = F_e g$

$\therefore \boxed{\mathcal{W}(g) = \frac{1}{2}\frac{q^2}{\epsilon A} g}$



For a capacitor C:

$q = CV \rightarrow V = \frac{q}{C}$

$\therefore \mathcal{W}(q) = \int_0^q V dq = \int_0^q \left(\frac{q}{C}\right) dq$

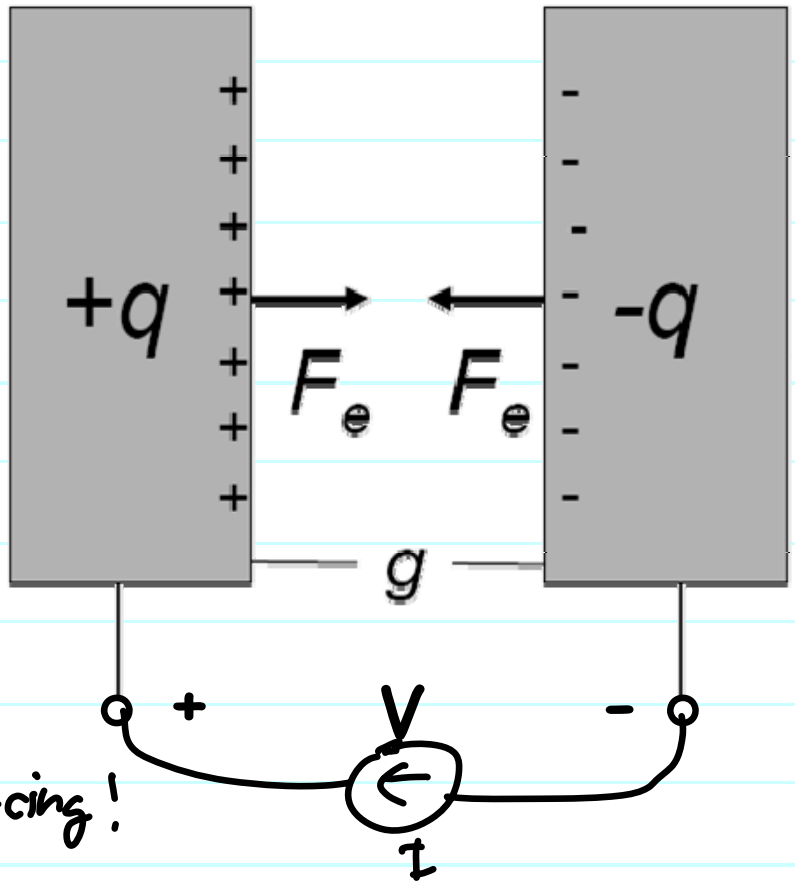
$= \frac{1}{2}\frac{q^2}{C} = \boxed{\frac{1}{2}\frac{q^2 g}{\epsilon A} = \mathcal{W}(q)}$

Work done to charge C to q at fixed gap

same answer when

Charge-Control Case

- Having found stored energy, we can now find the force acting on the plates and the voltage across them:



From $dW = V dq + F_e dg$

$\Rightarrow$ Force is given by

$$F_e = \left.\frac{\partial W(q,g)}{\partial g}\right|_{q=\text{const.}} = \frac{\partial}{\partial g}\left(\frac{1}{2}\frac{q^2}{\epsilon A} g\right)$$

$$\therefore \boxed{F_e = \frac{1}{2}\frac{q^2}{\epsilon A}}$$ $\Rightarrow$ independent of gap spacing!

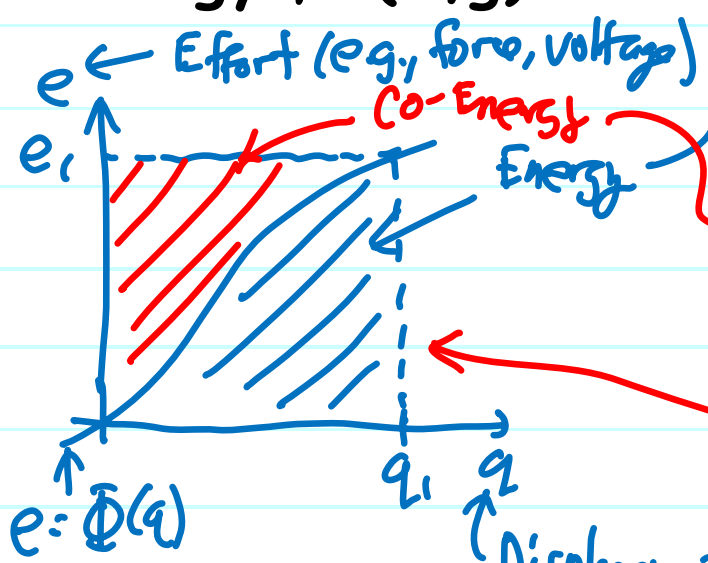
$\Rightarrow$ Voltage is given by,

$$V = \left.\frac{\partial W(q,g)}{\partial q}\right|_{g=\text{const.}} = \frac{\partial}{\partial q}\left(\frac{1}{2}\frac{q^2}{\epsilon A} g\right) = \frac{qg}{\epsilon A} \therefore \boxed{V = \frac{q}{C}}$$

$\Rightarrow$ consistent w/ what we already know! ✓

Voltage-Control Case

- Practical situation: We control V
  - Charge control on the typical sub-pF MEMS actuation capacitor is difficult
  - Need to find F_e as a partial derivative of the stored energy $\mathcal{W} = \mathcal{W}(V,g)$ with respect to g with V held constant? But can't do this with present $\mathcal{W}(q,g)$ formula
  - <u>Solution</u>: Apply Legendre transformation and define the co-energy $\mathcal{W}'(V,g)$



$$\mathcal{W}(q_1) = \int_0^{q_1} e\,dq = \int_0^{q_1} \Phi(q)\,dq$$

$$\mathcal{W}'(e_1) = \int_0^{e_1} q\,de = \int_0^{e_1} \Phi^{-1}(e)\,de$$

For a linear system, these will be equal

$\Rightarrow$ Can define co-energy as: $\mathcal{W}'(e) = eq - \mathcal{W}(q)$ (from this pl.d)

Co-Energy Formulation

- For our present problem (i.e., movable capacitive plates), the co-energy formulation becomes

$$\mathcal{W}'(V,g) = qV - \mathcal{W}$$

Differentially, this becomes:

$$d\mathcal{W}'(V,g) = (qdV + Vdq) - d\mathcal{W}(q,g)$$

But $[d\mathcal{W}(q,g) = F_e dg + Vdq]$

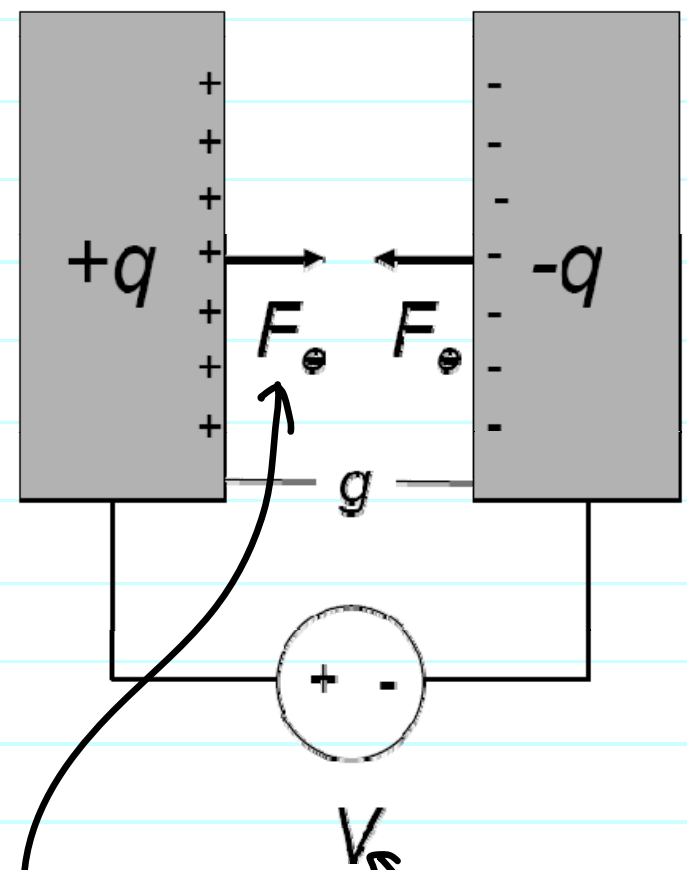
$$\boxed{d\mathcal{W}'(V,g) = qdV - F_e dg}$$

Working Co-Energy Expression

From which:

Charge, $q = \left.\dfrac{\partial \mathcal{W}'(V,g)}{\partial V}\right|_{g=\text{const.}}$

Force, $F_e = -\left.\dfrac{\partial \mathcal{W}'(V,g)}{\partial g}\right|_{V=\text{const.}}$ ⇒ this gives force as a function of applied voltage



Electrostatic Force (Voltage Control)

- Find co-energy in terms of voltage

$$W' = \int_0^V q(g,V')dV' = \int_0^V \left(\varepsilon \frac{A}{g}\right) V' dV' = \frac{1}{2}\left(\frac{\varepsilon A}{g}\right) V^2 = \frac{1}{2}CV^2$$

(as expected)

- Variation of co-energy with respect to gap yields electrostatic force:

$$F_e = -\frac{\partial W'(V,g)}{\partial g}\bigg|_V = -\frac{1}{2}\underbrace{\left(-\frac{\varepsilon A}{g^2}\right)}V^2 = \frac{1}{2}\frac{C}{g}V^2$$

strong function of gap!

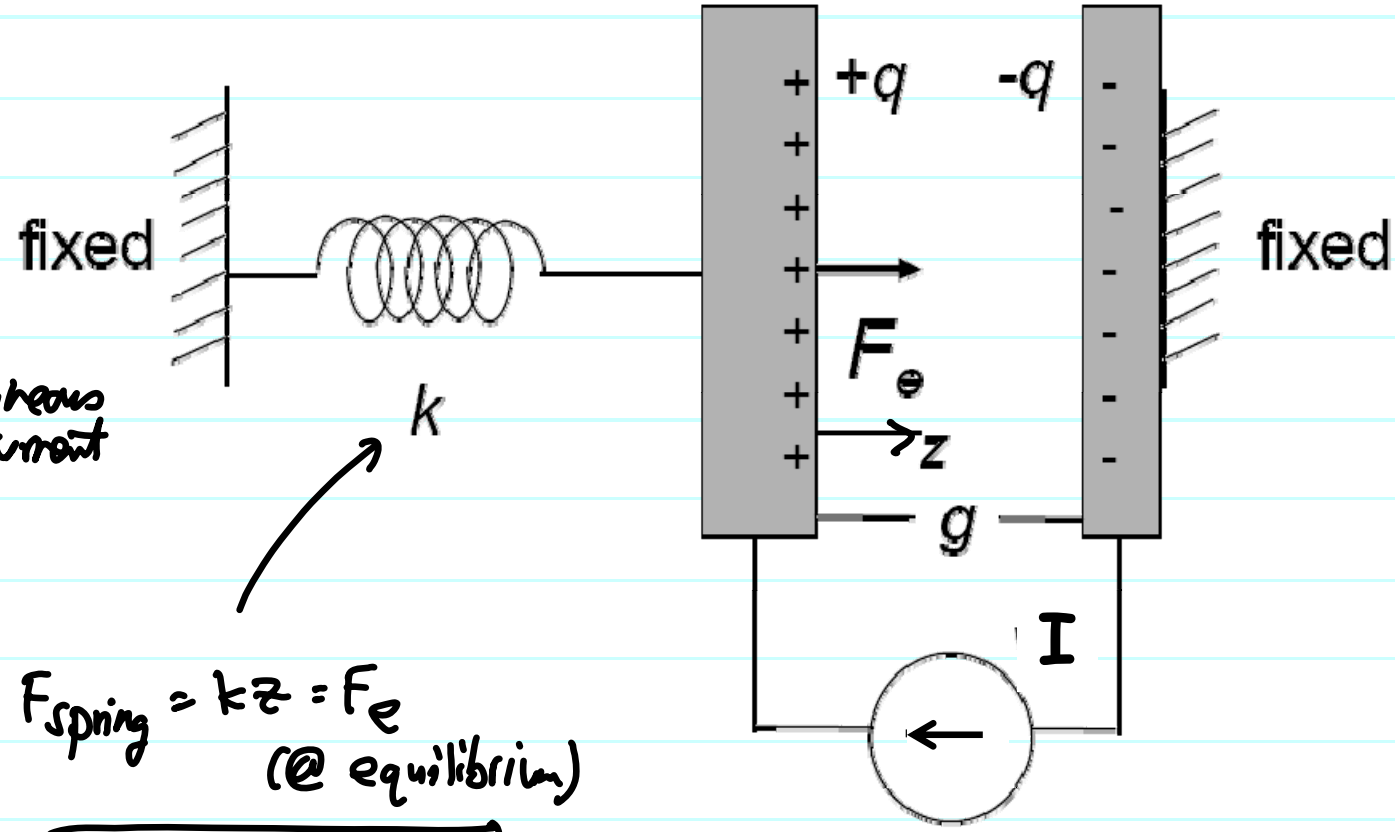
- Variation of co-energy with respect to voltage yields charge:

$$q = \frac{\partial W'(V,g)}{\partial V}\bigg|_g = \left(\frac{\varepsilon A}{g}\right)V = CV$$

as expected

Spring-Suspended Capacitive Plate

Charge Control of a Spring-Suspended C



Force generated by charge q supplied by current I: ← an instantaneous current

$$F_e = \left.\frac{\partial W(q,g)}{\partial g}\right|_q = \frac{q^2}{2\epsilon A}$$

Restoring force of spring: $F_{spring} = kz = F_e$ (@ equilibrium)

And the gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k} = g}$$

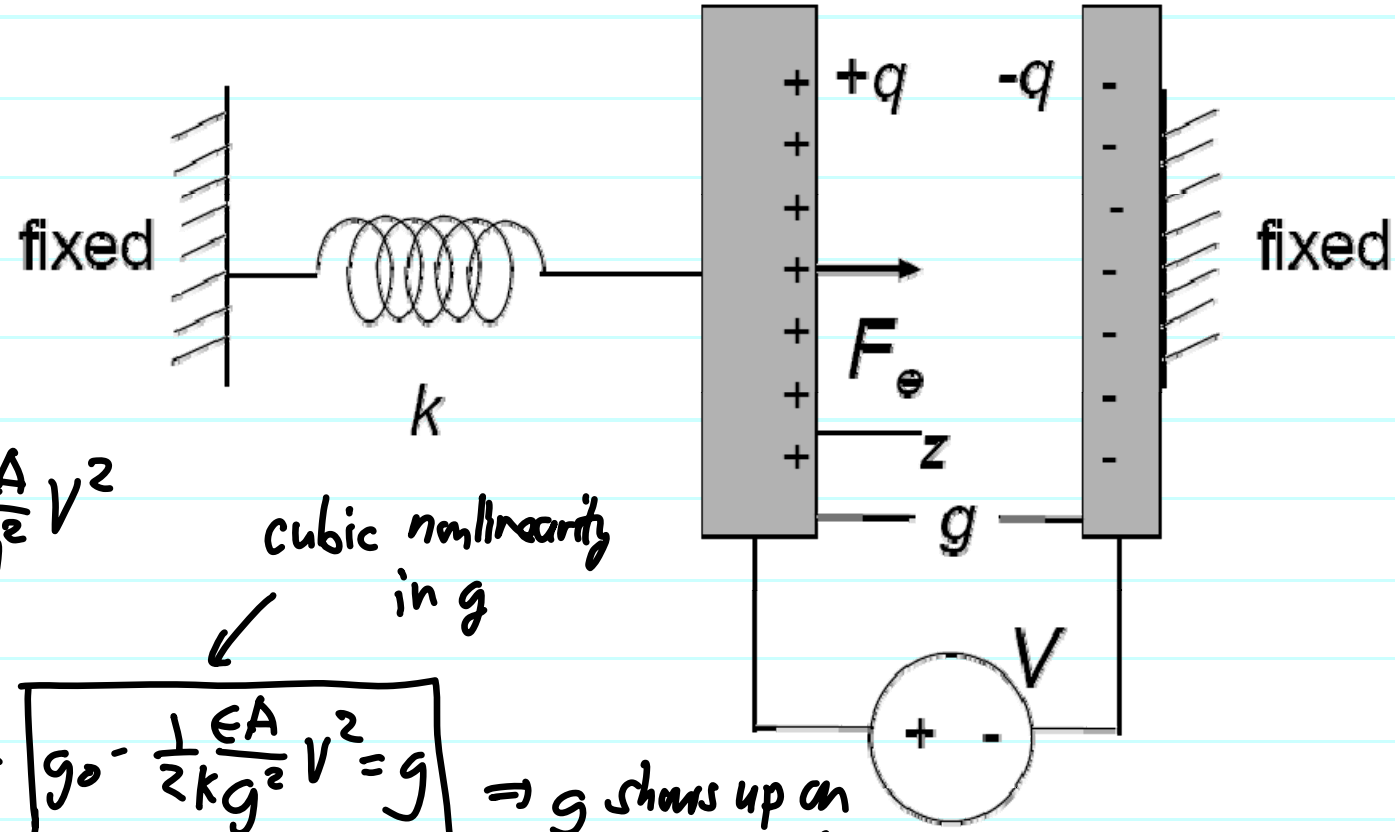
g_0 ← initial gap

⇒ can increase q and drive $g \to 0$

$$V = \frac{q}{C} = \frac{q}{\epsilon A} g = \boxed{\frac{q}{\epsilon A}\left(g_0 - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k}\right) = V}$$

⇒ V↓ as g↓

Voltage Control of a Spring-Suspended C



Again, $F_{spring} = kz = F_e$

But now:

$$F_e = \left.\frac{\partial W'(V,g)}{\partial g}\right|_q = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{\epsilon A}{kg^2}V^2 = g}$$

cubic nonlinearity in g

$\Rightarrow$ g shows up on both sides!

Charge: (for a stable gap)

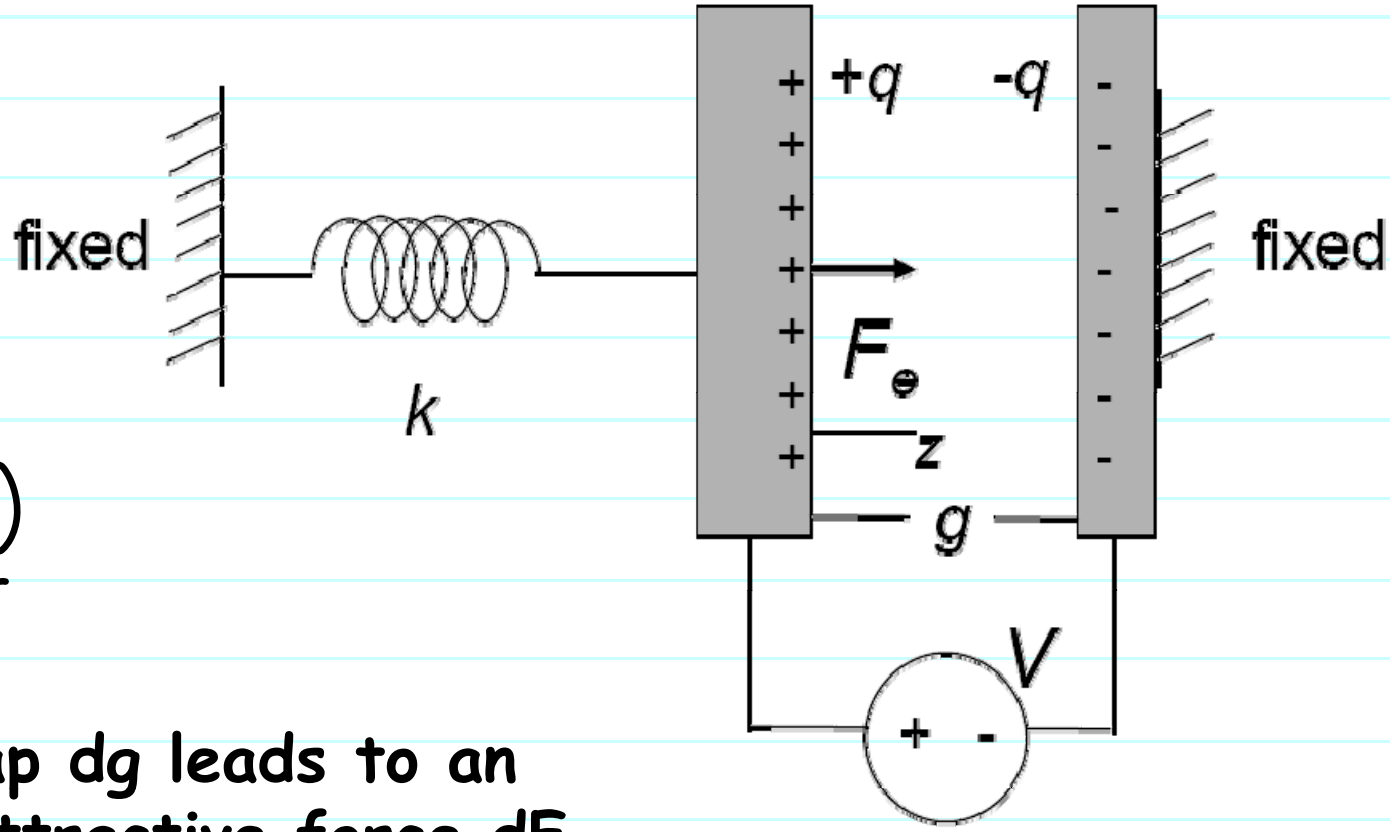
$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_g = CV = \boxed{\frac{\epsilon A}{g}V = q}$$

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$

(+) Feedback!

If loop gain > 1, then this will go unstable!

Stability Analysis



- Net attractive force on the plate:

$$F_{net} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_0 - g)}_{F_{spring}}$$

- An increment in gap dg leads to an increment in net attractive force dF_{net}

$$dF_{net} = \frac{\partial F_{net}}{\partial g} dg = \left[-\frac{\epsilon A V^2}{g^3} + k \right] dg$$

Thus, need this = (+) $\Rightarrow$ $\boxed{k > \frac{\epsilon A V^2}{g^3}}$

(for a stable uncollapsed state)

If $g\downarrow \rightarrow dg = (-)$

then for stability need $\rightarrow$ otherwise, plate collapses

$F_{net}\downarrow \rightarrow dF_{net} = (-)$

Pull-In Voltage V_{PI}

- V_{PI} = voltage at which the plates collapse
- The plate goes unstable when

$$k = \frac{\varepsilon A V_{PI}^2}{g_{PI}^3} \quad (1) \qquad \text{and} \qquad F_{net} = 0 = \frac{\varepsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_o - g_{PI}) \quad (2)$$

pull-in gap

- Substituting (1) into (2):

$$0 = \frac{\cancel{\varepsilon A V_{PI}^2}}{2 g_{PI}^{\cancel{2}}} - \frac{\cancel{\varepsilon A V_{PI}^2}}{g_{PI}^{\cancel{3}}}(g_o - g_{PI})$$

$$\frac{g_o - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_o = \frac{3}{2} g_{PI}$$

$$\therefore \boxed{g_{PI} = \frac{2}{3} g_o}$$

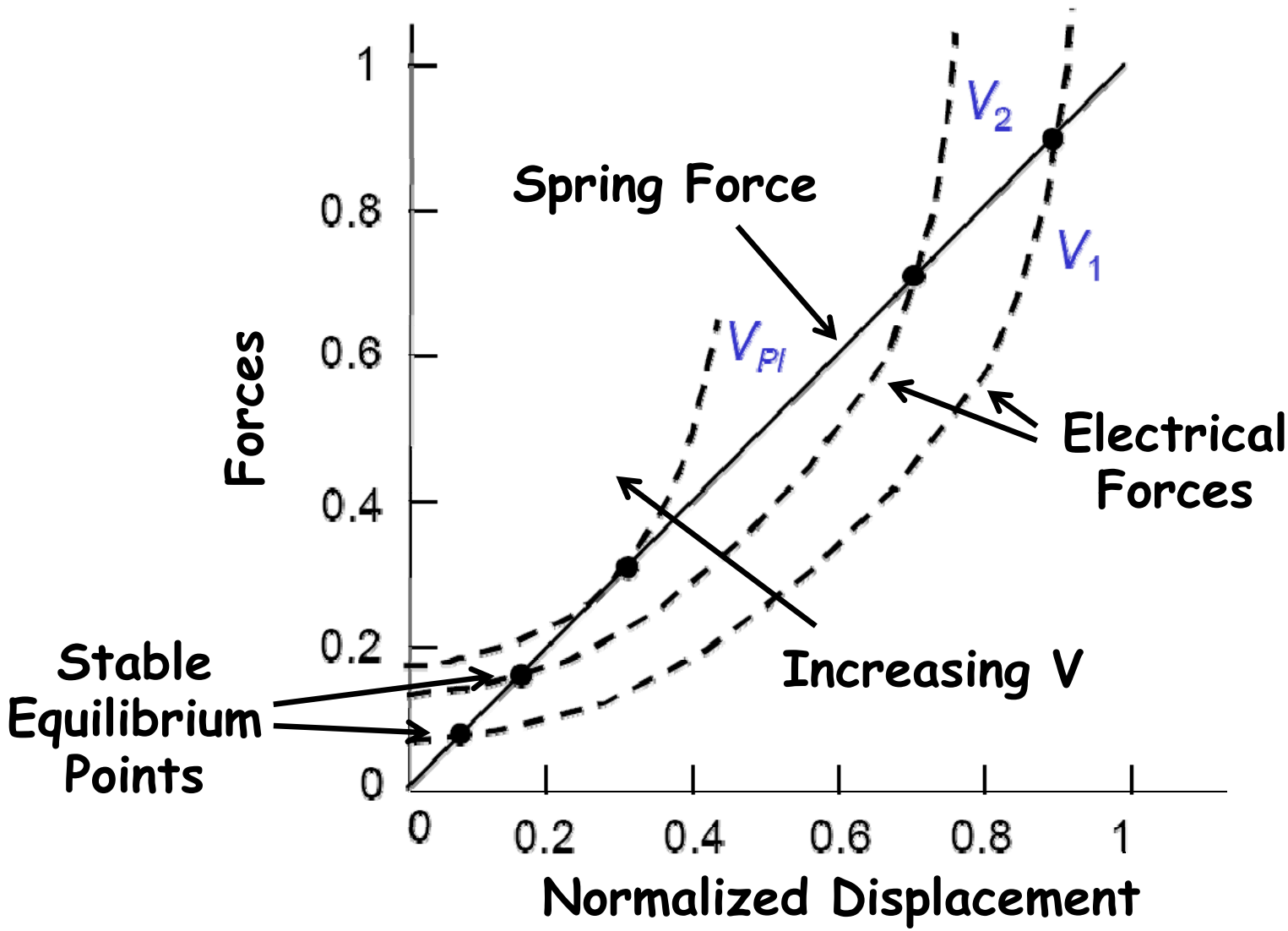
$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}}$$

$$\therefore \boxed{V_{PI} = \sqrt{\frac{8}{27} \frac{k g_o^3}{\epsilon A}}}$$

When a gap is driven by a voltage to (2/3) its original spacing, collapse will occur!

Voltage-Controlled Plate Stability Graph

- <u>Below</u>: Plot of normalized electrostatic and spring forces vs. normalized displacement $1-(g/g_o)$



Advantages of Electrostatic Actuators

- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

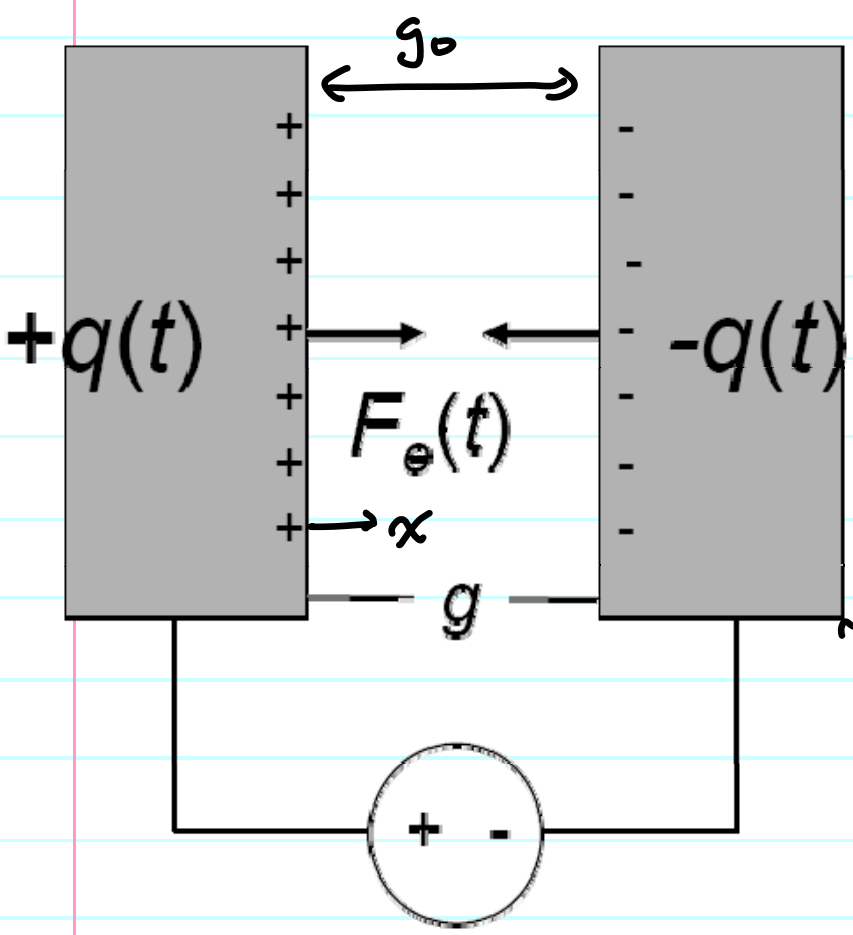
Problems With Electrostatic Actuators

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale

Linearizing the Voltage-to-Force Transfer Function

Linearizing the Voltage-to-Force T.F.

- Apply a DC bias (or polarization) voltage V_P together with the intended input (or drive) voltage $v_i(t)$, where $V_P >> v_i(t)$



$$v(t) = V_P + v_i(t)$$

$$F_e(t) = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2} C [v(t)]^2\right)$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}[v(t)]^2 = \frac{1}{2}(V_P + v_i(t))^2 \frac{\partial C}{\partial x}$$

$$= \frac{1}{2}\left[V_P^2 + 2V_P v_i(t) + \cancel{[v_i(t)]^2}\right]\frac{\partial C}{\partial x}$$

$$[V_P >> v_i(t)] \Rightarrow F_e(t) = \underbrace{\frac{1}{2}V_P^2 \frac{\partial C}{\partial x}}_{\text{DC Offset}} + \underbrace{V_P \frac{\partial C}{\partial x} v_i(t)}_{\text{AC Drive Signal}}$$

linear

$V_P \sim 10V$, $|v_i| \sim 10mV$

$$C(x) = \frac{\epsilon A}{g_o - x} = C_o\left(1 - \frac{x}{g_o}\right)^{-1} \approx C_o\left(1 + \frac{x}{g_o}\right)$$

$\frac{\epsilon A}{g_o} = C_o$

$[x << g_o] \longrightarrow$ ~linear

$$\therefore \frac{\partial C}{\partial x} \approx \frac{C_o}{g_o} \Rightarrow \boxed{F_e(t) = \frac{1}{2}\frac{C_o}{g_o}V_P^2 + V_P \frac{C_o}{g_o} v_i(t)}$$