Lecture 22: Capacitive Transducers
Logistics

• All EE’s to one side; ME’s to the other
• Email me your group makeups
• Makeup/Extra Lecture: During next Monday’s discussion
Lecture Outline

• Reading: Senturia, Chpt. 5, Chpt. 6

• Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
Energy Conserving Transducers
Basic Physics of Electrostatic Actuation

- **Goal**: Determine gap spacing $g$ as a function of input variables.
- First, need to determine the energy of the system.
- Two ways to change the energy:
  - Change the charge $q$
  - Change the separation $g$

$$\Delta W(q,g) = V \Delta q + F_e \Delta g$$

$$dW = Vdq + F_edg$$

- **Note**: We assume that the plates are supported elastically, so they don't collapse.
• Here, the stored energy is the work done in increasing the gap after charging capacitor at zero gap

\[ W = 0 + \int_0^g F_e \, dg' \]

\[ F_e = \left( \frac{q}{2} \right) \varepsilon = \frac{1}{2} \frac{q^2}{\varepsilon A} \]

(independent of \( g \))

\[ W = \left[ F_e g' \right]_0^g = F_e g \]

\[ W(g) = \frac{1}{2} \frac{q^2}{\varepsilon A} g \]

No change in charge: \( dq = 0 \)

\[ W(q, g) \rightarrow \text{zero gap} \rightarrow \text{zero stored energy} \]

For a capacitor \( C \):

\[ q = CV \rightarrow V = \frac{q}{C} \]

\[ W(q) = \int_0^q V \, dq = \int_0^q \left( \frac{q}{C} \right) \, dq = \frac{1}{2} \frac{q^2}{C} = \frac{1}{2} \frac{q^2}{\varepsilon A} = W(q) \]
• Having found stored energy, we can now find the force acting on the plates and the voltage across them:

\[ W = Vd_q + F_e d_g \]

- Force is given by:

\[ F_e = \frac{\partial W(q,g)}{\partial g} \bigg|_{q=\text{cont.}} \cdot \frac{\partial}{\partial g} \left( \frac{1}{2} \frac{q^2}{\epsilon A} g \right) \]

\[ F_e = \frac{1}{2} \frac{q^2}{\epsilon A} \]

\[ \Rightarrow \text{independent of gap spacing!} \]

- Voltage is given by:

\[ V = \frac{\partial W(q,g)}{\partial q} \bigg|_{g=\text{cont.}} \cdot \frac{\partial}{\partial q} \left( \frac{1}{2} \frac{q^2}{\epsilon A} g \right) : \frac{q q}{\epsilon A} \]

\[ V = \frac{q}{\epsilon} \]

\[ \Rightarrow \text{consistent with what we already know!} \]
Voltage-Control Case

- Practical situation: We control $V$
  - Charge control on the typical sub-pF MEMS actuation capacitor is difficult
  - Need to find $F_e$ as a partial derivative of the stored energy $W = W(V,g)$ with respect to $g$ with $V$ held constant? But can’t do this with present $W(q,g)$ formula

Solution: Apply Legendre transformation and define the co-energy $W'(V,g)$

\[
W'(e) = \int_0^e qde = \int_0^e x'(e)de
\]

(Equation for a linear system, these will be equal)

Can define co-energy as: $W'(e) = eq - W(q)$ (from this plot)
Co-Energy Formulation

• For our present problem (i.e., movable capacitive plates), the co-energy formulation becomes

\[ W'(V, g) = qV - W \]

Differentially, this becomes:

\[ dW'(V, g) = (qdv + Vdg) - dW(q, g) \]

But \[ dW(q, g) = F_0 dg + Vdg \]

\[ dW'(V, g) = qdv - F_0dg \]

From which:

Charge, \( q = \frac{\partial W'(V, g)}{\partial V} \) \( \mid V: \text{const.} \)

Force, \( F_e = -\frac{\partial W'(V, g)}{\partial g} \) \( \mid V: \text{const.} \)

= this gives force as a function of applied voltage
• Find co-energy in terms of voltage:

\[
W' = \int_{0}^{V} q(g, V') dV' = \int_{0}^{V} \left( \varepsilon \frac{A}{g} \right) V' dV' = \frac{1}{2} \left( \frac{\varepsilon A}{g} \right) V^2 = \frac{1}{2} CV^2
\]

(as expected)

• Variation of co-energy with respect to gap yields electrostatic force:

\[
F_e = -\frac{\partial W'(V, g)}{\partial g} = -\frac{1}{2} \left( -\frac{\varepsilon A}{g^2} \right) V^2 = \frac{1}{2} \frac{C}{g} V^2
\]

strong function of gap!

• Variation of co-energy with respect to voltage yields charge:

\[
q = \frac{\partial W'(V, g)}{\partial V} \bigg|_g = \left( \frac{\varepsilon A}{g} \right) V = CV
\]

as expected
Spring-Suspended Capacitive Plate
Charge Control of a Spring-Suspended C

Force generated by charge \( q \) supplied by current \( I \) at an instantaneous current

\[
F_e = \frac{\partial W(q, q)}{\partial q} \bigg|_{q} = \frac{q^2}{2 \varepsilon A}
\]

Restoring force of spring: \( F_{\text{spring}} = k \varepsilon = F_e \) (at equilibrium)

And the gap:

\[
g = g_0 - \frac{z}{k} = g_0 - \frac{F_e}{k}
\]

Initial gap

\[
V = \frac{q}{C} = \frac{q}{\varepsilon A} g = \frac{q}{\varepsilon A} \left( g_0 - \frac{1}{2} \frac{q^2}{\varepsilon A} \frac{1}{k} \right) = \nu
\]

\[\Rightarrow \text{can increase } q \text{ and drive } g \to 0\]

\[\Rightarrow \text{v} \downarrow \text{as } g \downarrow\]
Voltage Control of a Spring-Suspended C

Again, \( F_{spring} = k \frac{z^2}{2} \) \( F_e \)

But now:
\[
F_e = \frac{dW(V,g)}{dg} \bigg|_{g} = \frac{1}{2} \frac{\epsilon A}{g^2} V^2
\]

And \( k \) gap:
\[
g = g_0 - z = g_0 - \frac{F_e}{k} = g_0 - \frac{1}{2} \frac{\epsilon A}{k g^2} V^2 = g
\]

\( g \) shows up on both sides!

Charge: (for a stable gap)
\[
q = \frac{dW'(V,g)}{dV} \bigg|_{g} = CV = \frac{\epsilon A}{g} V = q
\]

If \( V \rightarrow g \rightarrow F_e \)?

(4) Feedback!

If loop gain > 1, then this will go unstable!
• Net attractive force on the plate:

\[ F_{\text{net}} = \frac{\varepsilon AV^2}{2g^2} - k(g_0 - g) \]

• An increment in gap \( dg \) leads to an increment in net attractive force \( dF_{\text{net}} \):

\[ dF_{\text{net}} = \frac{\partial F_{\text{net}}}{\partial g} \, dg = \left[ -\frac{\varepsilon AV^2}{g^3} + k \right] \, dg \]

Thus, need this = (4) \( k > \frac{\varepsilon AV^2}{g_0^3} \)

If \( g_0 \rightarrow g_0 = (\varepsilon) \)

then \( F_{\text{net}} \rightarrow \) otherwise, plate collapses
Pull-In Voltage $V_{PI}$

- $V_{PI}$ = voltage at which the plates collapse
- The plate goes unstable when
  \[
  k = \frac{\varepsilon A V_{PI}^2}{g_{PI}} \quad (1)
  \]
  \[
  F_{net} = 0 = \frac{\varepsilon A V_{PI}^2}{2 g_{PI}} - k (g_0 - g_{PI}) \quad (2)
  \]
- Substituting (1) into (2):
  \[
  0 = \frac{\varepsilon A V_{PI}^2}{2 g_{PI}^2} - \frac{\varepsilon A V_{PI}^2}{g_{PI}^2} (g_0 - g_{PI})
  \]
  \[
  \frac{g_0 - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_0 = \frac{3}{2} g_{PI}
  \]
  \[
  \therefore g_{PI} = \frac{2}{3} g_0
  \]

When a gap is driven by a voltage to (2/3) its original spacing, collapse will occur!
• **Below:** Plot of normalized electrostatic and spring forces vs. normalized displacement $1-(g/g_0)$
Advantages of Electrostatic Actuators

• Easy to manufacture in micromachining processes, since conductors and air gaps are all that’s needed → low cost!

• Energy conserving → only parasitic energy loss through $I^2R$ losses in conductors and interconnects

• Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement

• Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!

• Same capacitive structures can be used for both drive and sense of velocity or displacement

• Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q’s for resonant structures
Problems With Electrostatic Actuators

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale
Linearizing the Voltage-to-Force Transfer Function
Linearizing the Voltage-to-Force T.F.

- Apply a DC bias (or polarization) voltage $V_p$ together with the intended input (or drive) voltage $v_i(t)$, where $V_p \gg v_i(t)$

$$v(t) = V_p + v_i(t)$$

$$F_v(t) = \frac{\partial W'}{\partial x} = \frac{3}{2} C \left[ V_n(t) \right]^2$$

$$= \frac{1}{2} \frac{\partial C}{\partial x} \frac{\left[ V_n(t) \right]^2}{2} = \frac{1}{2} \left( V_p + V_n(t) \right)^2 \frac{\partial C}{\partial x}$$

$$= \frac{1}{2} \left( V_p^2 + 2V_p V_n(t) + V_n(t)^2 \right) \frac{\partial C}{\partial x}$$

$$\left[ V_p \gg V_n(t) \right] \Rightarrow F_v(t) = \frac{1}{2} V_p^2 \frac{\partial C}{\partial x} + V_p \frac{\partial C}{\partial x} V_n(t)$$

**Diagram:**

- Input voltage $v_i(t)$
- Output voltage $v(t)$
- DC bias $V_p$
- Capacitance $C$
- Acceleration $g$

**Equations:**

$$C(x) = \frac{EA}{g_0 x} = C_0 \left( 1 - \frac{x}{g_0} \right)^{-1} \approx C_0 \left( 1 + \frac{x}{g_0} \right)$$

$$\frac{\partial C}{\partial x} = C_0$$

$$\frac{2x}{g_0} \approx C_0 \Rightarrow F_v(t) = \frac{1}{2} \frac{C_0}{g_0} V_p^2 + V_p \frac{C_0}{g_0} V_n(t)$$