Lecture 23: Electrical Stiffness
Lecture Outline

• Reading: Senturia, Chpt. 5, Chpt. 6

• Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
Linearizing the Voltage-to-Force Transfer Function
Linearizing the Voltage-to-Force T.F.

- Apply a DC bias (or polarization) voltage $V_p$ together with the intended input (or drive) voltage $v_i(t)$, where $V_p >> v_i(t)$

$$v(t) = V_p + v_i(t)$$

$$F_e(t) = \frac{\partial W}{\partial \chi} \cdot \frac{\partial}{\partial \chi} \left( \frac{1}{2} C \left[ V_i(t) \right]^2 \right)$$

$$= \frac{1}{2} \frac{\partial C}{\partial \chi} \left[ V_i(t) \right]^2 = \frac{1}{2} \left( V_p + V_i(t) \right)^2 \frac{\partial C}{\partial \chi}$$

$$= \frac{1}{2} \left[ V_p^2 + 2V_p V_i(t) + \left[ V_i(t) \right]^2 \right] \frac{\partial C}{\partial \chi}$$

$$\left( V_p >> V_i(t) \right) \Rightarrow F_e(t) = \frac{1}{2} V_p^2 \frac{\partial C}{\partial \chi} + V_p \frac{\partial C}{\partial \chi} V_i(t)$$

$C(\chi) = \frac{EA}{g_0 \cdot \chi} = C_0 \left( 1 - \frac{\chi}{g_0} \right)^{-1} \approx C_0 \left( 1 + \frac{\chi}{g_0} \right)$

$$\left( \frac{\chi}{g_0} < C \right) \Rightarrow F_e(t) = \frac{1}{2} \frac{C_0}{g_0} V_p^2 + V_p \frac{C_0}{g_0} V_i(t)$$
• The net force on the suspended center electrode is

\[ F_{net} = F_{er}(t) - F_{el}(t) \]

\[
F_{net}(t) = \frac{1}{2} \frac{\partial C}{\partial x} \left\{ [N_R(t)]^2 - [N_L(t)]^2 \right\}
\]

\[
= \frac{1}{2} \frac{\partial C}{\partial x} \left\{ (V_p^2 + 2V_p N(t) + [N(t)]^2) - (V_p^2 - 2V_p N(t) + [N(t)]^2) \right\}
\]

\[
\therefore F_{net}(t) = 2V_p \frac{\partial C}{\partial x} N(t) = 2V_p \frac{C_0}{g_0} N(t)
\]

\[ \Rightarrow \text{linear w/ } N(t) \text{ ! (gap match limited)} \]
Remaining Nonlinearity (Electrical Stiffness)
Parallel-Plate Capacitive Nonlinearity

- **Example**: clamped-clamped laterally driven beam with balanced electrodes

- **Nomenclature**:
  
  \[ V_a \text{ or } v_A \]
  \[ v_a = |v_a| \cos \omega t \]
  \[ V_A \]
  \[ V_a \text{ or } v_A = V_A + v_a \]
  \[ V_1 \text{ or } v_1 \]
  \[ V_P \]
  
  **Conductive Structure**
  \[ k_m \]
  \[ d_1 \]
  \[ m \]
  \[ x \]
  
  **Total Value**
  **AC or Signal Component** (lower case variable; lower case subscript)
  **DC Component** (upper case variable; upper case subscript)
Parallel-Plate Capacitive Nonlinearity

- **Example**: clamped-clamped laterally driven beam with balanced electrodes
- **Expression for \( \frac{\partial C}{\partial x} \):
• Thus, the expression for force from the left side becomes:
Parallel-Plate Capacitive Nonlinearity

• Retaining only terms at the drive frequency:

\[ F_{d1} \mid_{\omega_o} = V_{P1} \frac{C_{o1}}{d_1} |v_1| \cos \omega_o t + V_{P1}^2 \frac{C_{o1}}{d_1^2} |x| \sin \omega_o t \]

Drive force arising from the input excitation voltage at the frequency of this voltage

Proportional to displacement

90° phase-shifted from drive, so in phase with displacement

• These two together mean that this force acts against the spring restoring force!

\[ A \text{ negative spring constant} \]

\[ A \text{ Since it derives from } V_p, \text{ we call it the electrical stiffness, given by:} \]

\[ k_e = V_{P1}^2 \frac{C_{o1}}{d_1^2} = V_{P1}^2 \frac{\varepsilon A}{d_1^3} \]
Electrical Stiffness, $k_e$

- The electrical stiffness $k_e$ behaves like any other stiffness.
- It affects resonance frequency:

$$\omega'_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_m - k_e}{m}}$$

$$= \sqrt{\frac{k_m}{m} \left(1 - \frac{k_e}{k_m}\right)^{1/2}}$$

$$\omega'_o = \omega_o \left(1 - \frac{V_{P1}^2 \varepsilon A}{k_m d_1^3}\right)^{1/2}$$

Frequency is now a function of dc-bias $V_{P1}$.
Voltage-Controllable Center Frequency

Quadrature force voltage-controllable electrical stiffness:

\[ k_e = \frac{\varepsilon_0 A_0 V_p^2}{d^3} \]

\[ f_0 = \frac{1}{2\pi} \sqrt{\frac{k_m - k_e}{m_r}} \]

Graph showing frequency as a function of DC bias voltage, with a peak shift of 1.1%.
Microresonator Thermal Stability

-1.7ppm/°C

Poly-Si μresonator - 17ppm/°C

* Thermal stability of poly-Si micromechanical resonator is 10X worse than the worst case of AT-cut quartz crystal
**Geometric-Stress Compensation**

- Use a temperature dependent mechanical stiffness to null frequency shifts due to Young’s modulus thermal dep.

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**Problems:**
- Stress relaxation
- Compromised design flexibility

[Hsu et al, IEDM’00]
Voltage-Controllable Center Frequency

$V_P$

*Quadrature force $\Rightarrow$ voltage-controllable electrical stiffness:*

$$k_e = \frac{\varepsilon_0 A_o V^2}{d^3}$$

$$f_o = \frac{1}{2\pi} \sqrt{\frac{k_m - k_e}{m_r}}$$

- **Electrode Overlap Area**
- **Gap**

- **Micromechanical Resonator**
- **Anchor**
- **Electrode**
- **Silicon Nitride**
- **Isolation Oxide**
- **Silicon Substrate**

- **Figure:**
  - Frequency vs. DC-Blas ($V_P$)
  - $A_o = 88 \mu m^2$
  - $d = 1000 \AA$
  - 1.1% change at $V_P = 20$
Excellent Temperature Stability

- Top Electrode-to-Resonator Gap ↑
- Elect. Stiffness: $k_e \sim 1/d^3$ ↓
- Frequency: $f_o \sim (k_m - k_e)^{0.5}$ ↑
- Counteracts reduction in frequency due to Young’s modulus temp. dependence

- AT-cut Quartz Crystal at Various Cut Angles
- On par with quartz!

- Elect.-Stiffness Compensation $-0.24$ ppm/oC
- $-1.7$ ppm/oC

Uncompensated $\mu$resonator

[Ref: Hafner]

[Ref: Hsu et al MEMS’02]
Measured $\Delta f/f$ vs. $T$ for $k_e$-Compensated $\mu$Resonators

**Design/Performance:**

- $f_o=10\text{MHz}$, $Q=4,000$
- $V_P=8\text{V}$, $h_e=4\mu\text{m}$
- $d_o=1000\text{Å}$, $h=2\mu\text{m}$
- $W_r=8\mu\text{m}$, $L_r=40\mu\text{m}$

[Hsu et al MEMS’02]

- Slits help to release the stress generated by lateral thermal expansion ⇒ linear $TC_f$ curves ⇒ $-0.24\text{ppm/°C}$!
Can One Cancel $k_e$ w/ Two Electrodes?

- What if we don’t like the dependence of frequency on $V_p$?
- Can we cancel $k_e$ via a differential input electrode configuration?
- If we do a similar analysis for $F_{d2}$ at Electrode 2:

  \[
  F_{d2}\bigg|_{\omega_o} = -V_{P2} \frac{C_{o2}}{d_2} |v_2| \cos \omega_o t \\
  + V_{P2}^2 \frac{C_{o2}}{d_2^2} |x| \sin \omega_o t
  \]

Subtracts from the $F_{d1}$ term, as expected

Adds to the quadrature term $\rightarrow k_e$’s add, no matter the electrode configuration!
Problems With Parallel-Plate C Drive

- Nonlinear voltage-to-force transfer function
  - Resonance frequency becomes dependent on parameters (e.g., bias voltage $V_p$)
  - Output current will also take on nonlinear characteristics as amplitude grows (i.e., as $x$ approaches $d_o$)
  - Noise can alias due to nonlinearity

- Range of motion is small
  - For larger motion, need larger gap ... but larger gap weakens the electrostatic force
  - Large motion is often needed (e.g., by gyroscopes, vibromotors, optical MEMS)
Electrostatic Comb Drive
• Use of comb-capacitive transducers brings many benefits
  ➔ Linearizes voltage-generated input forces
  ➔ (Ideally) eliminates dependence of frequency on dc-bias
  ➔ Allows a large range of motion
Comb-Drive Force Equation (1st Pass)

Top View

Side View

\[ d \]

\[ L_f \]

\[ h \]

Shuttle Finger

Drive Finger

\[ V_P \]

\[ V_i \]
Lateral Comb-Drive Electrical Stiffness

- Again: \( C(x) = \frac{2N\varepsilon_h x}{d} \rightarrow \frac{\partial C}{\partial x} = \frac{2N\varepsilon_h}{d} \)

- No \((\partial C/\partial x)\) \(x\)-dependence \(\rightarrow\) no electrical stiffness: \(k_e = 0\)!
- Frequency immune to changes in \(V_P\) or gap spacing!
Typical Drive & Sense Configuration

2-port Lateral Microresonator

\[ N_f : \text{# shuttle fingers} \]

**Simple Analysis:**

\[
  F_{d1} = \frac{1}{2} \frac{\partial C_1}{\partial x} (V_i - V_{P1})^2 = \frac{1}{2} \left( -\frac{\varepsilon_0 h}{d_1} \left( V_i^2 - 2V_{P1}V_i + V_{P1}^2 \right) \right) (2N_f)
\]

\[
  F_{d2} = \frac{1}{2} \frac{\partial C_2}{\partial x} (V_2 - V_{P2})^2 = \frac{1}{2} \left( \frac{\varepsilon_0 h}{d_2} \left( V_2^2 - 2V_{P2}V_2 + V_{P2}^2 \right) \right) (2N_f)
\]

\[
  F_{\text{net}} = F_{d1} + F_{d2} = \frac{1}{2} \left( \frac{\varepsilon_0 h}{d_1} \left( V_i^2 - V_2^2 - 2(V_{P2}V_2 - V_{P1}V_i) + V_{P2}^2 - V_{P1}^2 \right) \right) (2N_f)
\]
• In our 1st pass, we neglected
  Fringing fields
  Parallel-plate capacitance between stator and rotor
  Capacitance to the substrate

• All of these capacitors must be included when evaluating the energy expression!
Comb-Drive Force With Ground Plane Correction

- Finger displacement changes not only the capacitance between stator and rotor, but also between these structures and the ground plane → modifies the capacitive energy

\[
F_{e,x} = \frac{\partial W'}{\partial x} = \frac{1}{2} \frac{dC_{sp}}{dx} V_s^2 + \frac{1}{2} \frac{dC_{rp}}{dx} V_r^2 + \frac{1}{2} \frac{dC_{rs}}{dx} (V_s - V_r)^2
\]

[Gary Fedder, Ph.D., UC Berkeley, 1994]
Capacitance Expressions

• Case: \( V_r = V_p = 0V \)

• \( C_{sp} \) depends on whether or not fingers are engaged

\[
C'_{sp} = N[C'_{sp,e} x + C'_{sp,u} (L - x)]
\]

\[
C_{rs} = NC'_{rs} x
\]

Capacitance per unit length

Region 2

Region 3

[Gary Fedder, Ph.D., UC Berkeley, 1994]
Comb-Drive Force With Ground Plane Correction

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\]

\[
F_{e,x} = \frac{N}{2} \left( C_{rs}^l + C_{sp,e}^l - C_{sp,u}^l \right) V_s^2
\]

(for \( V_r = V_p = 0 \))

[Gary Fedder, Ph.D., UC Berkeley, 1994]
Simulate to Get Capacitors → Force

- Below: 2D finite element simulation

\[ F_{e,x} = \frac{N}{2} \left( C_{rs} + C_{sp,e} - C_{sp,u} \right) V_s^2 \]

20-40% reduction of \( F_{e,x} \)

\[ w = t = g = z_0 = 2 \mu \text{m} \]
Vertical Force (Levitation)

\[ F_{e,z} = \frac{\partial W'}{\partial z} = \frac{1}{2} \frac{dC_{sp}}{dz} V_s^2 + \frac{1}{2} \frac{dC_{rp}}{dz} V_r^2 + \frac{1}{2} \frac{dC_{rs}}{dz} (V_s - V_r)^2 \]

• For \( V_r = 0 \text{V} \) (as shown):
  \[ F_{e,z} = \frac{1}{2} Nx \left[ \frac{d\left(C'_{sp,e} + C'_{rs}\right)}{dz} \right] V_s^2 \]
Simulated Levitation Force

- Below: simulated vertical force $F_z$ vs. $z$ at different $V_P$'s [f/ Bill Tang Ph.D., UCB, 1990]

  See that $F_z$ is roughly proportional to $-z$ for $z < z_0 \rightarrow$ it's like an electrical stiffness that adds to the mechanical stiffness

$$F_z \approx \gamma_z V_P^2 \frac{(z_0 - z)}{z_0} = k_e (z_0 - z)$$

Equilibrium levitation, $z_0$

Vertical levitation [µm]
Vertical Resonance Frequency

\[ \omega_z = \sqrt{\frac{k_z + k_e}{k_z}} \]

where \( k_e = \left( \frac{\gamma_z}{Z_0} \right) V^2 \)

- Vertical resonance frequency
- \( \omega_z / \omega_{zo} \)
- Lateral = resonance frequency
- Vertical resonance frequency at \( V_p = 0V \)

• Signs of electrical stiffnesses in MEMS:
  - Comb (x-axis) \( \rightarrow k_e = 0 \)
  - Comb (z-axis) \( \rightarrow k_e > 0 \)
  - Parallel Plate \( \rightarrow k_e < 0 \)
Suppressing Levitation

- Pattern ground plane polysilicon into differentially excited electrodes to minimize field lines terminating on top of comb
- **Penalty:** x-axis force is reduced
Force of Comb-Drive vs. Parallel-Plate

- Comb drive (x-direction)
  \[ V_1 = V_2 = V_S = 1V \]

\[ F_{e,x} = \frac{\varepsilon_0 t}{g} V_s^2 \]

- Differential Parallel-Plate (y-direction)
  \[ V_1 = 0V, V_2 = 1V \]

\[ F_{e,y} = \frac{1}{2} \frac{\varepsilon_0 t x}{g^2} V_2^2 \]

\[ \frac{F_{e,y}}{F_{e,x}} = \frac{\varepsilon_0 t x}{2 \varepsilon_0 t V_2^2} = \frac{1}{2} \frac{x}{g} \]

Parallel-plate generates a much larger force; but at the cost of linearity

Gap = \( g = 1 \ \mu m \),
Thickness = \( t = 2 \ \mu m \),
Finger length = \( L = 100 \ \mu m \),
Overlap length \( x = 75 \ \mu m \)