Lecture 23: Electrical Stiffness
Lecture Outline

- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
Linearizing the Voltage-to-Force Transfer Function
Linearizing the Voltage-to-Force T.F.

- Apply a DC bias (or polarization) voltage $V_p$ together with the intended input (or drive) voltage $v_i(t)$, where $V_p >> v_i(t)$

$$v(t) = V_p + v_i(t)$$
Differential Capacitive Transducer

The net force on the suspended center electrode is

\[ F_{net} = F_{er}(t) - F_{el}(t) \]

\[ F_{net}(t) = \frac{1}{2} \frac{\partial C}{\partial x} \left\{ \left[ N_R(t) \right]^2 - \left[ N_L(t) \right]^2 \right\} \]

\[ = \frac{1}{2} \frac{\partial C}{\partial x} \left\{ (V_p^2 + 2V_pN(t) + [N(t)]^2) - (V_p^2 - 2V_pN(t) + [N(t)]^2) \right\} \]

\[ \therefore F_{net}(t) = 2V_p \frac{\partial C}{\partial x} N(t) = 2V_p \frac{C_0}{g_0} N(t) \]

\( \Rightarrow \) Linear w/ \( N(t) \) (gap match limited)
Remaining Nonlinearity
(Electrical Stiffness)
Parallel-Plate Capacitive Nonlinearity

• Example: clamped-clamped laterally driven beam with balanced electrodes

• Nomenclature:

\[ V_a \text{ or } v_A \]

\[ v_a = |v_a| \cos \omega t \]

\[ V_A \]

\[ V_a \text{ or } v_A = V_A + v_a \]

Total Value

AC or Signal Component

DC Component

(lower case variable; lower case subscript)

(upper case variable; upper case subscript)
Parallel-Plate Capacitive Nonlinearity

- **Example**: clamped-clamped laterally driven beam with balanced electrodes

- Expression for $\frac{\partial C}{\partial x}$:
  \[
  C_i(x) = \frac{\varepsilon A}{d_1 + x} = C_0 i \left(1 + \frac{x}{d_1}\right)^{-1} \rightarrow \frac{\partial C_i}{\partial x} = -\frac{C_0 i}{d_1} \left(1 + \frac{x}{d_1}\right)^{-2}
  \]

  [Expand the Taylor series further]

  \[
  \frac{\partial C_i}{\partial x} = -\frac{C_0 i}{d_1} \left(1 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots \right)
  \]

  where
  \[
  A_1 = -\frac{2}{d_1},
  A_2 = \frac{3}{d_1^2},
  A_3 = -\frac{4}{d_1^3},
  \vdots
  \]
• Thus, the expression for force from the left side becomes:

\[ F_{d1} = \frac{1}{2} \frac{\partial C}{\partial x} (V_P - V_i - V_r)^2 = \frac{1}{2} \frac{\partial C}{\partial x} (V_{P1} - V_i)^2 \]

\[ = \frac{1}{2} (- \frac{C_0}{d_1})(1+A_1x)(V_{P1}^2 - 2V_{P1}V_i + V_i^2) \]

\[ = \frac{1}{2} (- \frac{C_0}{d_1}) \{ V_{P1}^2 - 2V_{P1}V_i + V_i^2 \} \]

@ resonance: \( x = \frac{Q F_{d1}}{jk} \approx \frac{Q}{jk} \frac{\partial C}{\partial x} V_{P1} V_i \)

Thus,

\( V_i = |V_i| \cos \omega t \rightarrow x \approx |x| \sin \omega t \)

\( x \) 90° phase-shifted from \( V_i \)
Parallel-Plate Capacitive Nonlinearity

- Retaining only terms at the drive frequency:

\[ F_{d1}|_{\omega_o} = V_{P1} \frac{C_{o1}}{d_1} |V_1| \cos \omega_o t + V_{P1}^2 \frac{C_{o1}}{d_1^2} |x| \sin \omega_o t \]

Drive force arising from the input excitation voltage at the frequency of this voltage

Proportional to displacement

90° phase-shifted from drive, so in phase with displacement

- These two together mean that this force acts against the spring restoring force!
  - A negative spring constant
  - Since it derives from \( V_p \), we call it the electrical stiffness, given by:

\[ k_e = V_{P1}^2 \frac{C_{o1}}{d_1^2} = V_{P1}^2 \frac{\varepsilon A}{d_1^3} \]
Electrical Stiffness, \( k_e \)

- The electrical stiffness \( k_e \) behaves like any other stiffness.
- It affects resonance frequency:

\[
\omega'_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_m - k_e}{m}}
\]

\[
= \sqrt{\frac{k_m}{m}} \left(1 - \frac{k_e}{k_m}\right)^{1/2}
\]

\[
\omega'_o = \omega_o \left(1 - \frac{V_{P1}^2 \varepsilon A}{k_m d_1^3}\right)^{1/2}
\]

Frequency is now a function of dc-bias \( V_{P1} \).
Voltage-Controllable Center Frequency

Quadrature force \( \Rightarrow \) voltage-controllable electrical stiffness:

\[
k_e = \frac{\varepsilon_0 A_o V^2}{d^3} \frac{1}{V_p}
\]

\[
f_o = \frac{1}{2\pi} \sqrt{\frac{k_m - k_e}{m_r}}
\]

Electrode Overlap Area

\( A_o = 88 \mu m^2 \)

\( d = 1000 \AA \)

1.1%
Microresonator Thermal Stability

-1.7ppm/°C

Poly-Si μresonator - 17ppm/°C

• Thermal stability of poly-Si micromechanical resonator is 10X worse than the worst case of AT-cut quartz crystal.
Geometric-Stress Compensation

- Use a temperature dependent mechanical stiffness to null frequency shifts due to Young's modulus thermal dep.

![Image of MEMS device with stress relaxation and compromised design flexibility](image)

Problems:
- Stress relaxation
- Compromised design flexibility

[Hsu et al, IEDM'00]
Voltage-Controllable Center Frequency

\[ V_P \]

**Quadrature force**

\[ k_e = \frac{\varepsilon_0 A_0 V^2}{d^3} \]

**Gap**

\[ f_0 = \frac{1}{2\pi} \sqrt{\frac{k_m - k_e}{m_r}} \]

Eletrode Overlap Area

**Frequency [MHz]**

- \( A_0 = 88\,\mu m^2 \)
- \( d = 1000\,\AA \)

1.1%
Excellent Temperature Stability

![Diagram showing temperature stability and resonator](image)

Top Electrode-to-Resonator Gap $\uparrow$

Elect. Stiffness: $k_e \approx 1/d^3$ $\downarrow$

Frequency: $f_0 \approx (k_m - k_e)^{0.5} \uparrow$

Counteracts reduction in frequency due to Young’s modulus temp. dependence

-1.7 ppm/°C

Elect.-Stiffness Compensation $-0.24$ ppm/°C

AT-cut Quartz Crystal at Various Cut Angles

Uncompensated $\mu$resonator

On par with quartz!

Ref: [Hafner]
Measured $\Delta f/f$ vs. $T$ for $k_e$-Compensated $\mu$Resonators

Design/Performance:

$f_0 = 10$MHz, $Q=4,000$

$V_P=8$V, $h_e=4\mu$m

$d_o=1000\AA$, $h=2\mu$m

$W_r=8\mu$m, $L_r=40\mu$m

[Hsu et al MEMS’02]

• Slits help to release the stress generated by lateral thermal expansion $\Rightarrow$ linear $TC_f$ curves $\Rightarrow$ $-0.24$ppm/°C!!!
Can One Cancel $k_e$ w/ Two Electrodes?

- What if we don't like the dependence of frequency on $V_p$?
- Can we cancel $k_e$ via a differential input electrode configuration?
- If we do a similar analysis for $F_{d2}$ at Electrode 2:

Subtracts from the $F_{d1}$ term, as expected:

$$F_{d2} \bigg|_{\omega_o} = -V_{P2} \frac{C_{o2}}{d_2} |v_2| \cos \omega_o t$$

$$+ V_{P2}^2 \frac{C_{o2}}{d_2^2} |x| \sin \omega_o t$$

Adds to the quadrature term $\rightarrow k_e$'s add, no matter the electrode configuration!
Problems With Parallel-Plate C Drive

- Nonlinear voltage-to-force transfer function
  - Resonance frequency becomes dependent on parameters (e.g., bias voltage $V_P$)
  - Output current will also take on nonlinear characteristics as amplitude grows (i.e., as $x$ approaches $d_o$)
  - Noise can alias due to nonlinearity

- Range of motion is small
  - For larger motion, need larger gap ... but larger gap weakens the electrostatic force
  - Large motion is often needed (e.g., by gyroscopes, vibromotors, optical MEMS)
Electrostatic Comb Drive
Electrostatic Comb Drive

- Use of comb-capacitive transducers brings many benefits
  - Linearizes voltage-generated input forces
  - (Ideally) eliminates dependence of frequency on dc-bias
  - Allows a large range of motion
Comb-Drive Force Equation (1st Pass)

\[ C(x) = \frac{2 \varepsilon e h}{d} \]

\[ \frac{\partial C}{\partial x} = \frac{2 \varepsilon e}{d} \]

\[ F_d = \frac{\partial W'}{\partial x} = \frac{1}{2} \frac{\partial C}{\partial x} (V_P - V_i)^2 = \frac{\varepsilon e}{2 d} (V_P^2 - 2V_P V_i + V_i^2) = -2V_P \frac{\varepsilon e}{d} V_i = F_d \]

But wait! This ignores some practical effects. (No dependence on x! LINEAR!)
Lateral Comb-Drive Electrical Stiffness

\[ C(x) = \frac{2N\varepsilon h x}{d} \rightarrow \frac{\partial C}{\partial x} = \frac{2N\varepsilon h}{d} \]

- Again: \( C(x) = \frac{2N\varepsilon h x}{d} \rightarrow \frac{\partial C}{\partial x} = \frac{2N\varepsilon h}{d} \)

- No (\( \frac{\partial C}{\partial x} \)) \( x \)-dependence \( \rightarrow \) no electrical stiffness: \( k_e = 0 \)
- Frequency immune to changes in \( V_p \) or gap spacing!
Typical Drive & Sense Configuration

2-port Lateral Microresonator

$N_f$: # shuttle fingers

Simple Analysis:

$$F_{d1} = \frac{1}{2} \frac{\partial C_1}{\partial x} (V_i - V_{p1})^2 = \frac{1}{2} \left( -\frac{\varepsilon_0 h}{d_1} \right) (V_i^2 - 2V_{p1}V_i + V_{p1}^2)(2N_f)$$

$$F_{d2} = \frac{1}{2} \frac{\partial C_2}{\partial x} (V_2 - V_{p2})^2 = \frac{1}{2} \left( \frac{\varepsilon_0 h}{d_2} \right) (V_2^2 - 2V_{p2}V_2 + V_{p2}^2)(2N_f)$$

$$F_{\text{net}} = F_{d1} + F_{d2} = \frac{1}{2} \left( \frac{\varepsilon_0 h}{d_1} \right) (V_i^2 - V_{p1}^2 - 2(V_{p2}V_2 - V_{p1}V_i) + V_{p2}^2 - V_2^2)(2N_f)$$

For $V_1 = V_2, V_i = -V_2$

$F_{\text{net}} = 2(2N_f) \left( \frac{\varepsilon_0 h}{d} \right) V_{p1}V_i$
• In our 1st pass, we neglected
  ➫ Fringing fields
  ➫ Parallel-plate capacitance between stator and rotor
  ➫ Capacitance to the substrate

• All of these capacitors must be included when evaluating the energy expression!
Comb-Drive Force With Ground Plane Correction

- Finger displacement changes not only the capacitance between stator and rotor, but also between these structures and the ground plane → modifies the capacitive energy

\[
F_{e,x} = \frac{\partial W'}{\partial x} = \frac{1}{2} \frac{dC_{sp}}{dx} V_s^2 + \frac{1}{2} \frac{dC_{rp}}{dx} V_r^2 + \frac{1}{2} \frac{dC_{rs}}{dx} (V_s - V_r)^2
\]

[Gary Fedder, Ph.D., UC Berkeley, 1994]
Capacitance Expressions

- **Case:** $V_r = V_p = 0V$
- $C_{sp}$ depends on whether or not fingers are engaged

\[
C'_{sp} = N[C'_{sp,e}x + C'_{sp,u}(L - x)]
\]

\[
C_{rs} = NC'_{rs}x
\]

Capacitance per unit length

Region 2

Region 3

[Gary Fedder, Ph.D., UC Berkeley, 1994]
Comb-Drive Force With Ground Plane Correction

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\[ F_{e,x} = \frac{\partial W'}{\partial x} = \frac{1}{2} \frac{dC_{sp}}{dx} V_s^2 + \frac{1}{2} \frac{dC_{rp}}{dx} V_r^2 + \frac{1}{2} \frac{dC_{rs}}{dx} (V_s - V_r)^2 \]

\[ F_{e,x} = \frac{N}{2} (C'_{rs} + C'_{sp,e} - C'_{sp,u}) V_s^2 \]

(for \( V_r = V_p = 0 \))

[Gary Fedder, Ph.D., UC Berkeley, 1994]
Simulate to Get Capacitors → Force

• Below: 2D finite element simulation

\[ F_{e,x} = \frac{N}{2}(C'_r + C'_{sp,e} - C'_{sp,u})V^2 \]

20-40% reduction of \( F_{e,x} \)