Lecture Outline

• Reading: Senturia, Chpt. 6, Chpt. 14

• Lecture Topics:
  - Input/Output Modeling
  - I/O Equivalent Circuit Models
    - Electromechanical Coupling
    - Mechanical Coupling
  - Detection Circuits
    - Position Sensing
    - Velocity Sensing
Complete Electrical-Port Equiv. Circuit

Static electrode-to-mass overlap capacitance

\[ F_{d1} \]

\[ k \]

\[ b \]

\[ l_x = m \]

\[ c_x = \frac{1}{k} \]

\[ r_x = b \]

\[ I_1 \]

\[ d_1 \]

\[ d_2 \]

\[ V_1 \]

\[ V_2 \]

\[ C_{o1} \]

\[ I_2 \]

\[ V_P \]

\[ \eta_{e1} = V_P \frac{\partial C_1}{\partial x} = V_P \frac{C_{o1}}{d_1} \]

\[ \eta_{e2} = V_P \frac{\partial C_2}{\partial x} = V_P \frac{C_{o2}}{d_2} \]
What is the impedance seen looking into port 1 with port 2 shorted to ground?

For our transformer model:

\[
\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ 0 & -\frac{i}{\eta} \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \end{bmatrix} \Rightarrow e_2 = \eta e_1, \quad e_1 = \frac{e_2}{\eta}
\]

\[
f_2 = -\frac{i}{\eta} f_1 \Rightarrow f_1 = -\eta f_2
\]
Input Impedance Into Port 2

What is the impedance seen looking into port 2 with port 1 shorted to ground?

\[
\frac{V_1}{I_{x2}} = 2\pi f \cdot \frac{1}{\eta_{e2}} \left( j\omega r_x + \frac{1}{j\omega C_x} + r_x \right) = j\omega \left( \frac{L_x}{\eta_{e2}} \right) + \frac{1}{j\omega (\eta_{e2} C_x)} + \frac{r_x}{\eta_{e2}^2}
\]

Note: These are not the same as \( L_{x1}, C_{x1}, R_{x1} \)!
What is the transconductance from port 1 to port 2 with port 2 shorted to ground?

\[
\dot{x} = \frac{1}{\eta_{e1}} \dot{i}_1
\]

\[
\dot{I}_0 = \frac{\dot{e}_2}{\eta_{e2}} \dot{x}
\]

\[
\begin{align*}
\dot{I}_0 &= \frac{\dot{e}_2}{\eta_{e2}} \dot{x} \\
\Rightarrow \quad \frac{\dot{I}_0}{N_a} &= \frac{1}{N_a} \cdot \frac{\dot{e}_2}{\eta_{e2}} \dot{x} \\
&= \frac{1}{N_a} \cdot \frac{\eta_{e1}\eta_{e2}}{\eta_{e1} + \eta_{e2}} \left( \frac{N_a}{2\pi} \right) \\
&= \frac{\eta_{e2}^2}{\eta_{e1}^2} \left( \frac{1}{\eta_{e1}} \right) \\
&= \frac{\eta_{e2}^2}{\eta_{e1}^2} \left( \frac{1}{\eta_{e1}} \right) \\
&= \frac{\eta_{e2}^2}{\eta_{e1}^2} \left( \frac{1}{\eta_{e1}} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{\dot{I}_0}{N_a} (j\omega) &= \frac{\eta_{e1}\eta_{e2}}{\eta_{e1} + \eta_{e2}} \left( \frac{N_a}{2\pi} \right) \\
&= \frac{\eta_{e2}^2}{\eta_{e1}^2} \left( \frac{1}{\eta_{e1}} \right) \\
&= \frac{\eta_{e2}^2}{\eta_{e1}^2} \left( \frac{1}{\eta_{e1}} \right)
\end{align*}
\]
Port 1 to 2 $v_i$-to-$i_o$ Transfer Function

\[
\frac{i_o(s)}{v_i(s)} = \frac{\eta e^{\frac{1}{ne}}}{\eta e^{\frac{1}{ne}} + j\omega l + \frac{1}{j\omega c_x + R_x}} = \left[ j\omega L_{x12} + \frac{j\omega C_{x12}}{s} + R_{x12} \right]^{-1} \begin{cases} 
L_{x12} = \frac{l_x}{\eta e^{\frac{1}{ne}}}
\end{cases}
\]

Separate freq. response & magnitude:

\[
\frac{i_o}{v_i}(s) = \frac{1}{s \frac{L_x}{C_x} + \frac{1}{s C_x} + R_x} = \frac{SC_x}{s^2 \frac{L_x}{C_x} + 1 + SC_x R_x} = \frac{s \left( \frac{1}{l_x} \right)}{s^2 + \frac{1}{L_x C_x} + s \left( \frac{R_x}{l_x} \right)}
\]

\[
\left[ \frac{1}{L_x C_x} = \omega_0^2, A = \frac{\omega L_x}{R_x} \rightarrow \frac{R_x}{L_x} = \frac{\omega_0}{A} \right] \Rightarrow \frac{i_o}{N_i}(s) = \frac{1}{R_x} \frac{s \omega_0 / A}{s^2 + s \omega_0 / A + \omega_0^2} = \frac{1}{R_x} \Theta(s)
\]

\[
\Theta(s) = \frac{s \omega_0 / A}{s^2 + s \omega_0 / A + \omega_0^2}
\]

\[
\begin{align*}
\Theta(0) & = 0 \\
\Theta(j \omega_0) & = 1 \\
\Theta(\infty) & = 0
\end{align*}
\]

Graph of $|\Theta(s)|$ and $\frac{i_o}{v_i}(s)$ with peak at $\omega_0$.

Gain Term (Bandpass Biquad) always the same.

Gain just do all calculations @ resonance $\omega_0$ just multiply by $\Theta(s)$. 
Condensed Equiv. Circuit (Symmetrical)

If $\eta_{e1} = \eta_{e2}$, then ...

$$L_x = \frac{m}{\eta_e^2}$$
$$C_x = \frac{\eta_e^2}{k}$$
$$R_x = \frac{b}{\eta_e^2}$$

Holds for the symmetrical case, where port 1 and port 2 are identical.
Phasings of Signals

- Below: plots of resonance electrical and mechanical signals vs. time, showing the phasings between them.
Sensing Circuits
MEMS-Based Tuning Fork Gyroscope

- Sense Electrodes
- Drive Voltage Signal
- Oscillation Sustaining Amplifier
- Differential TransR Sense Amplifier

[Zaman, Ayazi, et al, MEMS'06]
Detecting Velocity Versus Position

**Velocity**

- **Transfer Function:**

\[
\frac{\nu(s)}{F_d(s)} = \frac{1}{k} \frac{\omega_o^2 s}{s^2 + (\omega_o/Q)s + \omega_o^2}
\]

- Detect velocity when the output is at resonance or when a bandpass response is required

**Position**

(i.e., displacement)

- **Transfer Function:**

\[
\frac{X(s)}{F_d(s)} = \frac{1}{k} \frac{\omega_o^2}{s^2 + (\omega_o/Q)s + \omega_o^2}
\]

- Detect position when the output is varying slowly, i.e., at low frequencies
Detecting Velocity Versus Position

**Velocity**

• Transfer Function:

\[
\frac{\nu(s)}{F_d(s)} = \frac{1}{k} \frac{\omega_o^2 s}{s^2 + (\omega_o/Q)s + \omega_o^2}
\]

\[
= \frac{\omega_o Q}{k} \frac{(\omega_o/Q)s}{s^2 + (\omega_o/Q)s + \omega_o^2}
\]

\[
= \frac{\omega_o Q}{k} \mathcal{H}(s) \quad \text{Gain Bandpass Biquad Factor Freq. Response}
\]

\[
\therefore @ \omega_o: \frac{\nu}{F_d} = \frac{\omega_o Q}{k}
\]

**Position** (i.e., displacement)

• Transfer Function:

\[
\frac{X(s)}{F_d(s)} = \frac{1}{k} \frac{\omega_o^2}{s^2 + (\omega_o/Q)s + \omega_o^2}
\]

\[
= \frac{1}{k} \frac{(\omega_o/Q)s}{s^2 + (\omega_o/Q)s + \omega_o^2} \frac{\omega_o Q}{s}
\]

\[
= \frac{\omega_o Q}{k} \frac{1}{s} \mathcal{H}(s) \quad \text{Gain Bandpass Biquad Factor Freq. Response}
\]

\[
\int \mathcal{H}(s) ds = \frac{Q}{j\omega_o} \quad \text{Integrate}
\]

\[
@ \omega_o: \frac{X}{F_d} = \frac{\omega_o Q}{k} \frac{1}{j\omega_o} (1) = \frac{Q}{jk} \checkmark
\]
Output Current Measures Velocity

\[ I_1 \quad 1: \eta_{e_1} \quad l_x \quad c_x \quad r_x \quad \eta_{e_2}:1 \quad I_2 \]

\[ V_1 \quad C_{o_1} \quad V_2 \quad C_{o_2} \]

- Relationship between output current and velocity:

\[ i_o = \eta_{e_2} \dot{x} \]

Output current is proportional to velocity, and thus, directly measures velocity.

- To turn current into voltage (for a voltage output), send the current into a resistor \( R_L \).

- To get position, must integrate \( \rightarrow \) send the current into a capacitor \( C_L \).
To convert velocity to a voltage, use a resistive load.

\[ V_o = \frac{R_l}{N^2} \frac{\dot{\omega}}{\dot{\omega}}(s) \]

\[ \frac{N^2}{N^2} \frac{\dot{\omega}}{\dot{\omega}}(s) = \frac{\dot{\omega}}{\dot{\omega}}(s) \]

\[ \frac{R_l}{N^2} \frac{\dot{\omega}}{\dot{\omega}}(s) = \frac{\dot{\omega}}{\dot{\omega}}(s) \]

\[ Q = Q \left( \frac{R_l}{R_l + R_x} \right) \]