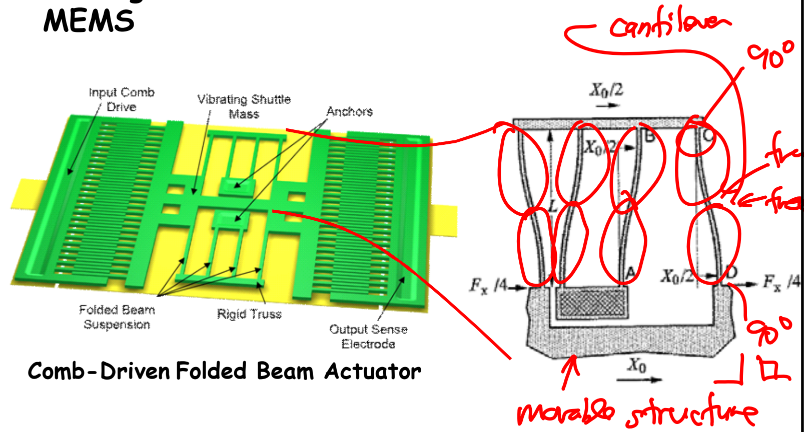
Lecture 14: Beam Bending

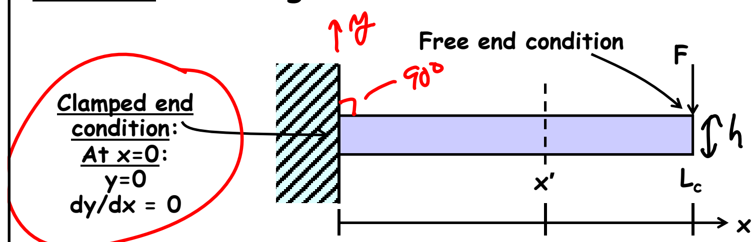
- Announcements:
- HW#4 online and due Tuesday, Oct 18
- Lecture Module 8 online
- Slight issue with HW#1 solutions
  - Some of the circuit schematics for superposition on the last problem were not quite right … but the solutions (i.e., equations and numbers) were still right
  - Fix is online
- Midterm is nearing: Thursday, Oct. 27
  - I will soon pass out materials associated with the midterm, including and information sheet and old exams
- Makeup Lecture:
  - I won't be here Thursday, next week
  - We will make up the lecture on Friday, 10/14, this week, in 2 LeConte, from 3-4:30 p.m.
- ------------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- ------------------------------------
- Last Time:
- Finished Module 7 on Mechanics of Materials
- Now start a new topic: Bending of Beams

- Springs and suspensions very common in MEMS
- Coils are popular in the macro-world; but not easy to make in the micro-world
- Beams: simpler to fabricate and analyze; become "stronger" on the micro-scale → use beams for MEMS



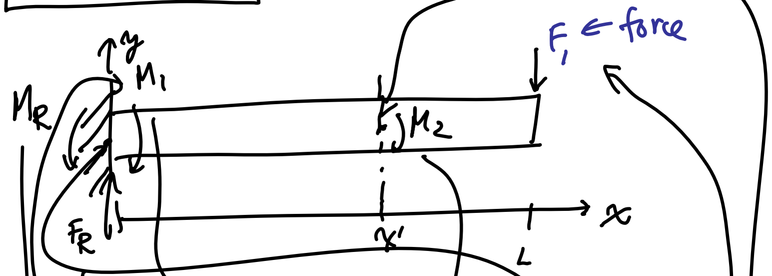
Comb-Driven Folded Beam Actuator

Problem: Bending a Cantilever Beam



- Objective: Find relation between tip deflection $y(x=L_c)$ and applied load F
- Assumptions:
  1. Tip deflection is small compared with beam length
  2. Plane sections (normal to beam's axis) remain plane and normal during bending, i.e., "pure bending"
  3. Shear stresses are negligible

Forces & Moments



Moment due to F_1 here $= M_1 = F_1 L$

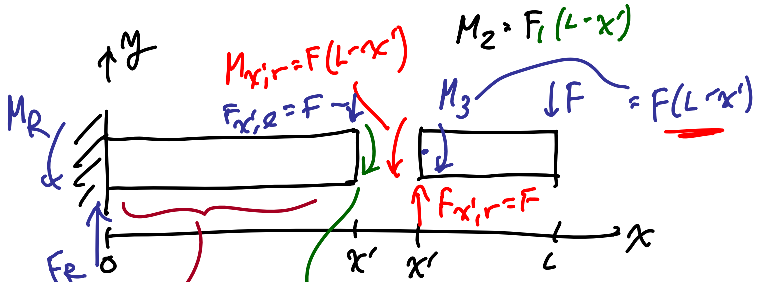
Reaction Force (to maintain equilibrium)

→ equal & opposite to

equal & opposite to

Moment due to F_1 here

$M_2 = F_1 (L - x')$



$M_{x',l} = M_{x',r} = F(L - x')$
↑ left

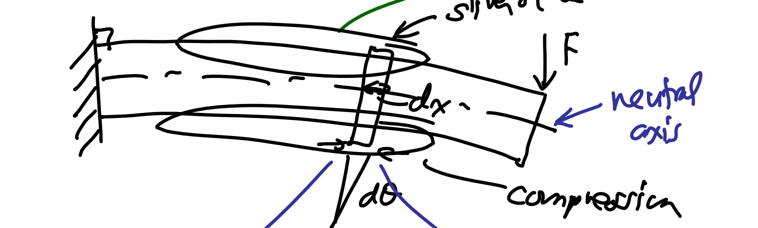
For the left side beam

Total Force: $F_R - F_{x',l} = F - F = 0$ ✓

Total Moment: $M_R - M_{x',l} + x' F_{x',l}$

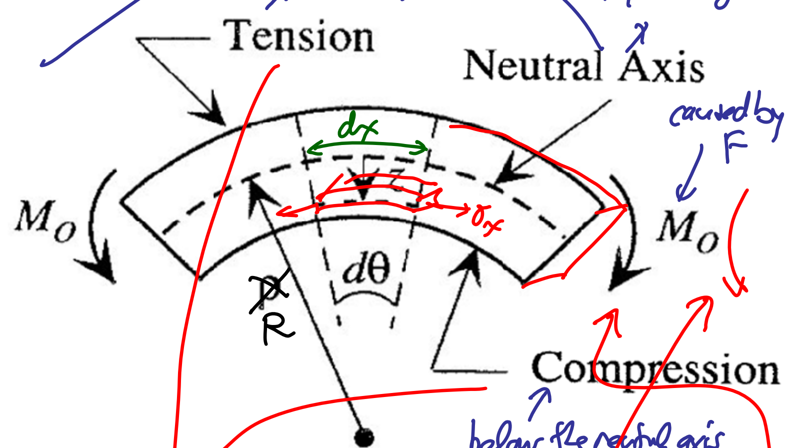
$= -FL + F(L - x') + x'F = 0$ ✓

Bent Beam (bent by applied force)



above the neutral axis

Length unchanged by bending



Combine to generate forces → moments that fight against the moment due to F

↓

This is how stiffness is generated!

↑

k_c → we want this!

Beam Segment in Pure Bending

⇒ consider the segment bounded by the dashed lines defining $d\theta$

At $z=0$: neutral axis → segment length $= dx = R\,d\theta$ (1)

At any z: segment length $= dL = (R-z)\,d\theta$ (2)

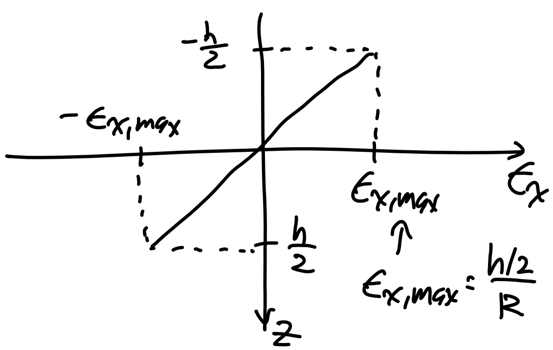
Combine (1) & (2): $dL = dx - z\,d\theta = dx - \frac{z}{R}dx$

Thus, the axial strain @ z:

$$\epsilon_x = \frac{dL - dx}{dx} = -\frac{z}{R}$$

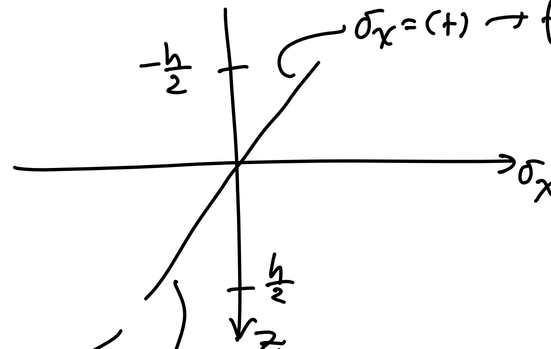
$$\boxed{\epsilon_x = -\frac{z}{R}}$$

Thus, the strain varies linearly along the beam thickness:



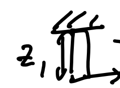
Of course, there is a corresponding axial stress:

$$\sigma_x = \epsilon_x E = \boxed{-\frac{zE}{R} = \sigma_x}$$

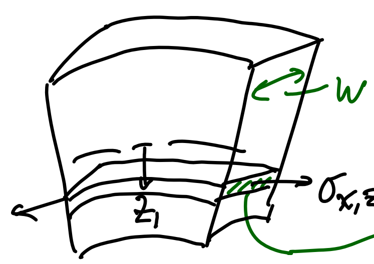


This gradient of stress then generates a bending moment! in response to the original applied moment (from F)

Stress → Force:



what is F?



$\sigma_{x,z_1} = \frac{F}{A} \rightarrow F = (W\,dz)\cdot\sigma_{x,z_1}$

$A = W\,dz$

⇒ integrate stress through the thickness of the beam:

$$M = \int_{-\frac{h}{2}}^{\frac{h}{2}} [\underbrace{(W dz)\sigma_x}_{\text{force}}]\cdot z$$

$$= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E W z^2}{R}\, dz \Rightarrow \boxed{M = -\left(\frac{1}{12}Wh^3\right)\frac{E}{R}}$$

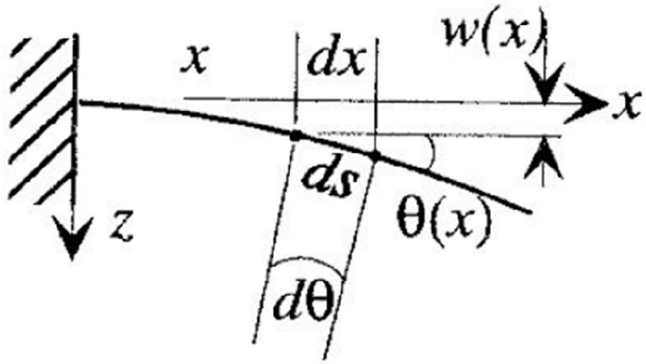
$\left[\sigma_x = -\frac{zE}{R}\right]$

$\frac{1}{12}Wh^3 = I \triangleq$ Moment of Inertia

$$\boxed{\frac{1}{R} = -\frac{M}{EI}}$$

Note: (+) of radius of curvature
↔ (−) internal bending moment

Differential Equation for Beam Bending



Write out some geometric relationships:

⇒ then use small angle approx:

$\cos\theta = \frac{dx}{ds} \rightarrow ds = \frac{dx}{\cos\theta} \longrightarrow ds \approx dx$

$\tan\theta = \frac{dw}{dx}$ = slope of the beam @ this point $\longrightarrow \theta \simeq \frac{dw}{dx}$ (1)

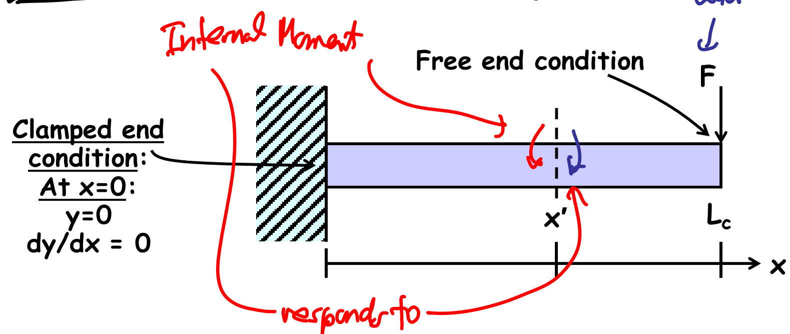
$ds = R\,d\theta \rightarrow \frac{1}{R} = \frac{d\theta}{ds} \longrightarrow \frac{1}{R} = \frac{d\theta}{dx}$ (2)

Inserting (1) into (2)

$$\frac{1}{R} = \boxed{\frac{d^2w}{dx^2} = -\frac{M}{EI}}$$

← Diff. Eqn. for Small Angle Beam Bending

$\boxed{\text{Cantilever Beam w/ Concentrated Load}}$



Internal Moment @ position x: $M = -F(L-x)$

Thus: $\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$

$$\text{w/} \begin{cases} \text{Clamped End B.C.: } w(x=0)=0,\ \dfrac{dw}{dx}(x=0)=0 \\ \text{Free End B.C.: none} \end{cases}$$