Lecture 15: Stress Gradients

- Announcements:
- Lecture Module 9 online
- Midterm is nearing: Thursday, Oct. 27
  - I will soon pass out materials associated with the midterm, including and information sheet and old exams
- Makeup Lecture:
  - I won't be here Thursday, next week
  - We will make up the lecture on Friday, 10/14, this week, in 2 LeConte, from 3-4:30 p.m.
- When turning in homework, staple all sheets together to avoid issues with lost pages
- --
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
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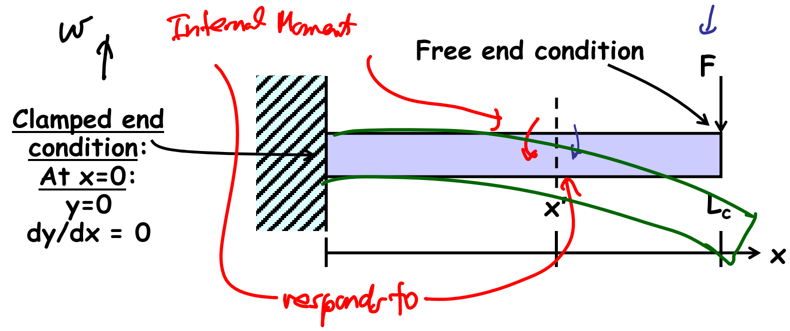
- Last Time:
- Beam bending theory

Cantilever Beam w/ Concentrated Load



Internal Moment @ position x: $M = -F(L-x)$

Thus: $\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$

w/ $\left\{ \begin{array}{l} \text{Clamped End B.C.: } w(x=0)=0,\ \frac{dw}{dx}(x=0)=0 \\ \text{Free End B.C.: None} \end{array} \right.$

Solve to get for w:

⇒ use Laplace; or a trial solution:

$w = A + Bx + Cx^2 + Dx^3$, then apply B.C.'s

$$\boxed{w = \frac{FL}{2EI}x^2\left(1 - \frac{x}{3L}\right)}$$

Deflection @ x due to a point load F applied @ $x=L$

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Maximum Deflection → occurs $x=L$:

$$w_{max} = \left(\frac{L^3}{3EI}\right)F \rightarrow F = \underbrace{\left(\frac{3EI}{L^3}\right)}_{\text{stiffness} \triangleq k_c} w(x=L) = k_c w(x=L)$$

where $k_c = \frac{3EI}{L^3} \triangleq$ stiffness @ $x=L$

in general, stiffness is a function of location

$$\left[I = \frac{1}{12}Wh^3\right] \rightarrow \boxed{k_c = \frac{1}{4}EW\frac{h^3}{L^3}}$$

Ex. $L = 100\mu m$, $W = 2\mu m$, $h = 2\mu m$

polysilicon → $E = 150$ GPa

$$k_c = \frac{1}{4}(150G)(2\mu)\left(\frac{2\mu}{100\mu}\right)^3 = 0.6 N/m$$

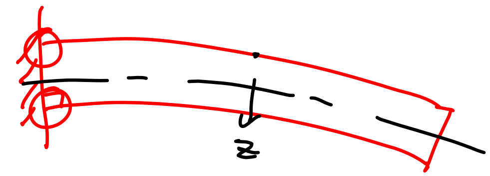
Maximum Stress in a Bent Cantilever

From before, the radius of curvature is given by

$$\frac{1}{R} = \frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

⇒ $\frac{1}{R}$ is maximized (i.e., R is minimized) when $x=0$:

$$[x=0] \rightarrow \frac{1}{R} = \frac{d^2w}{dx^2} = \frac{FL}{EI}$$ *



Strain is maximized:

① At top surface → tensile

② At bottom surface → compressive

$$\epsilon_{max} = \frac{z}{R} = \frac{h}{2}\frac{1}{R} = \frac{h}{2}\frac{FL}{EI} = \epsilon_{max}$$

*

$$\left[I = \frac{1}{12}Wh^3\right] \rightarrow \epsilon_{max} = \frac{\cancel{h}}{2}\frac{FL}{E}\left(\frac{12}{Wh^{\cancel{3}\,2}}\right) = \frac{6L}{EWh^2}F$$

$$\boxed{\sigma_{max} = \epsilon_{max}E = \frac{6L}{Wh^2}F}$$

(maximum stress in a bent cantilever subjected to a force F at its tip)

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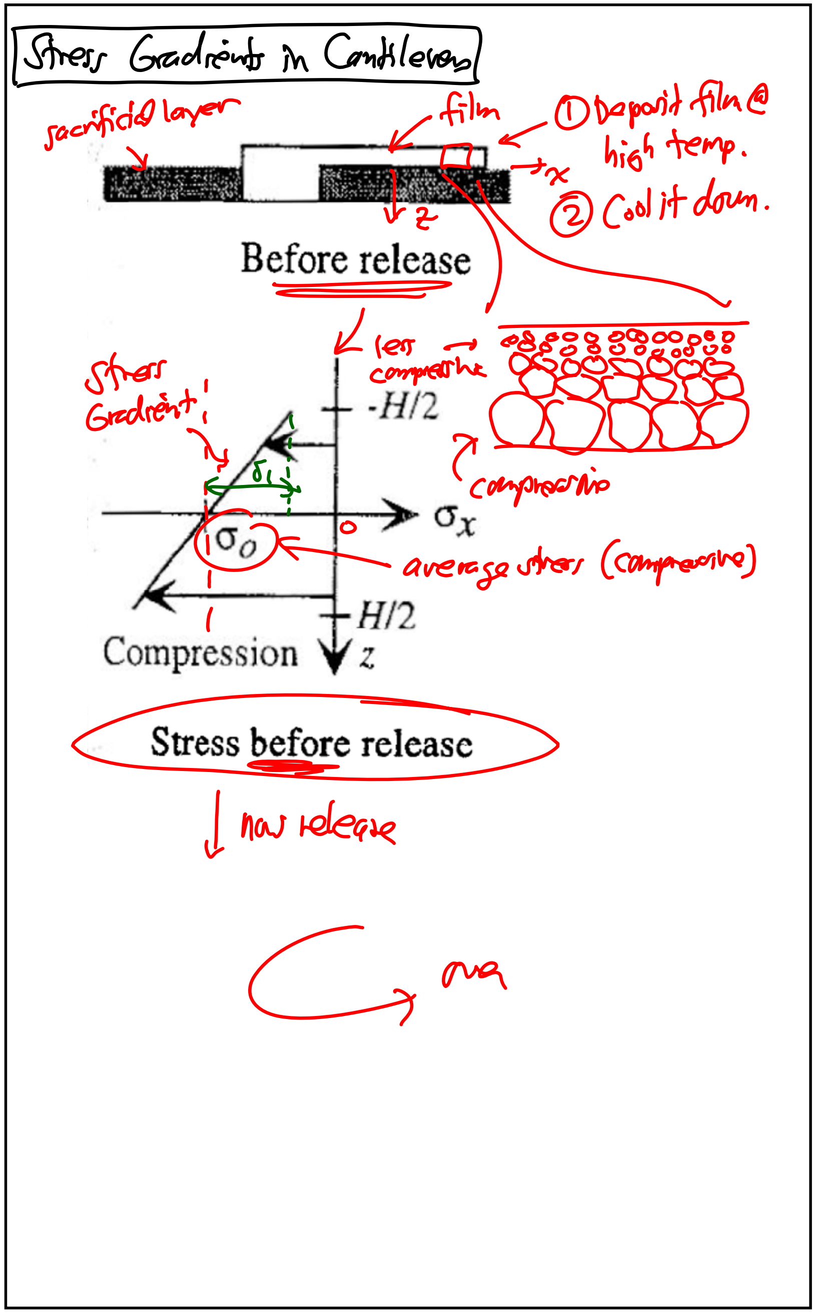

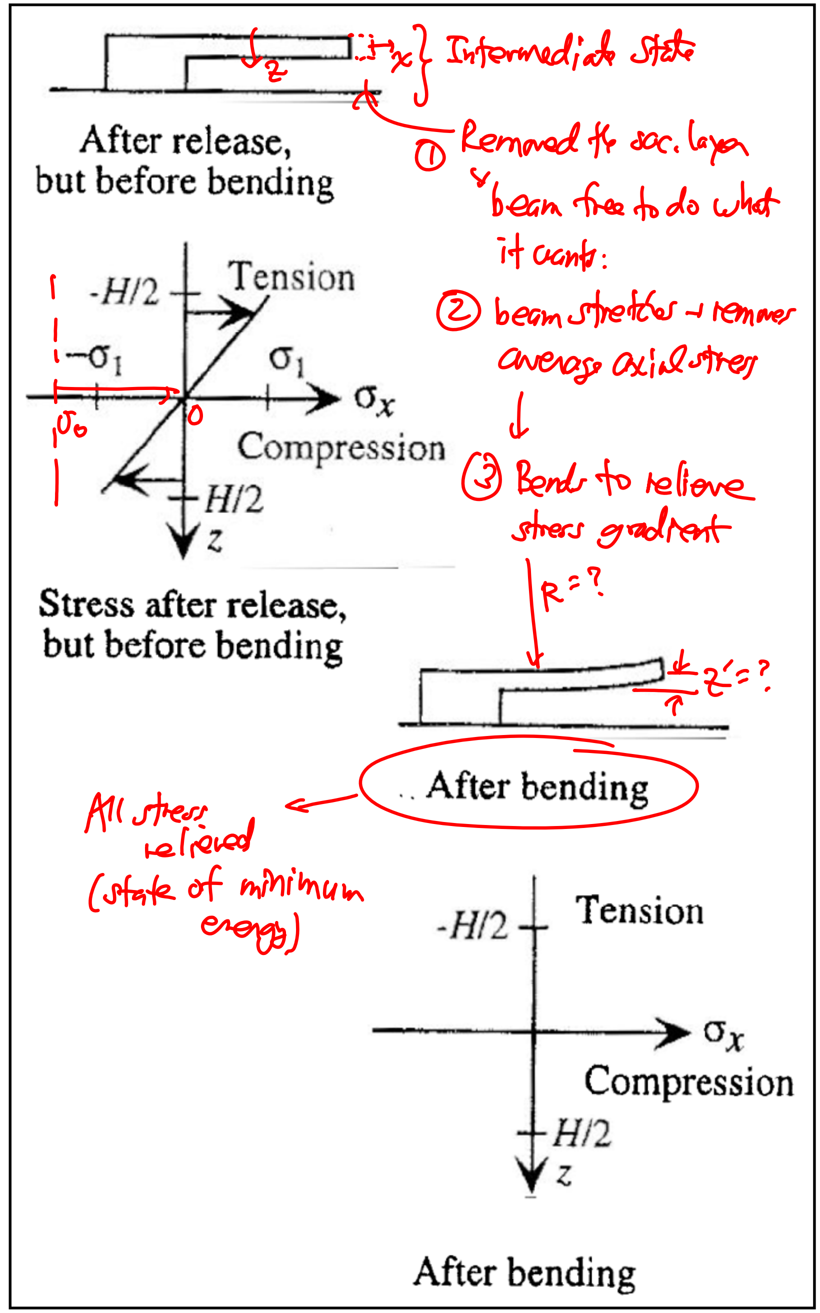

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Bending Due to Stress Gradients

Find the radius of curvature:

Prior to release, axial stress: $\sigma = \sigma_0 - \frac{\sigma_1}{(H/2)} z$

The internal moment:

$$M_x = \int_{-\frac{H}{2}}^{\frac{H}{2}} [(W \cdot dz)\cdot\sigma]\cdot z = \int_{-\frac{H}{2}}^{\frac{H}{2}} W\left(z\sigma_0 - \frac{\sigma_1 z^2}{(H/2)}\right) dz$$

$$= W\left(\frac{1}{2}\sigma_0 z^2 - \frac{2\sigma_1 z^3}{3H}\right)\Bigg|_{-H/2}^{H/2}$$

$$= W\left(\cancel{\frac{1}{2}\sigma_0 \frac{H^2}{4}} - \frac{2}{3}\sigma_1\frac{H^2}{8} + \cancel{\frac{1}{2}\sigma_0\frac{H^2}{4}} - \frac{2}{3}\frac{\sigma_1 H^2}{8}\right)$$

average stress cancels out

$$\boxed{M_x = -\frac{1}{6}\sigma_1 W H^2}$$

Thus, the radius of curvature:

$$\frac{1}{R} = -\frac{M_x}{E'I} \rightarrow R = \frac{E'I}{M_x} = \frac{1}{2}\frac{E'H}{\sigma_1}$$

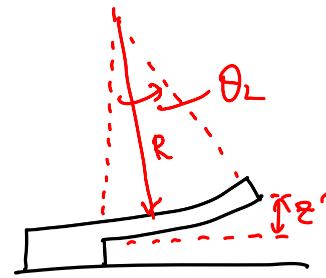
Biaxial Modulus

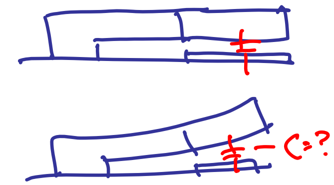
$$\left[I = \frac{1}{12} W h^3\right]$$

$$\boxed{R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1}}$$

Radius of Curvature for a Cantilever w/ a Stress Gradient

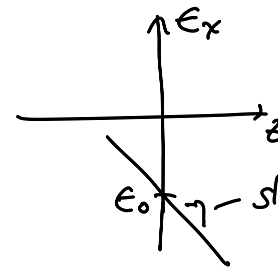
Radius of Curvature → z'





→ integrate over θ_L to get z'
↓
Do this is homework...

Definition: Stress Gradient



slope ≜ Strain Gradient = Γ

$$\Gamma = \frac{\epsilon_1}{(H/2)}$$

$$R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1} = \frac{H}{2}\frac{E'}{\sigma_1} = \frac{(H/2)}{\epsilon_1} = \frac{1}{\Gamma} \rightarrow \boxed{\Gamma = \frac{1}{R}}$$ ✓

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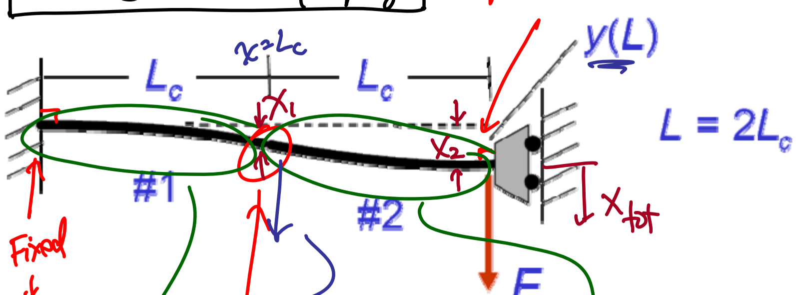
Series Combination of Springs



Fixed ↓ maintains 90°

Free (just like a cantilever)

cantilever → stiffness $= k_c$

also, a cantilever ↓ stiffness $= k_c$

Series: $x_{tot} > x_1$, $x_{tot} > x_2$ } $x_{tot} = x_1 + x_2$

What's the force law? → F

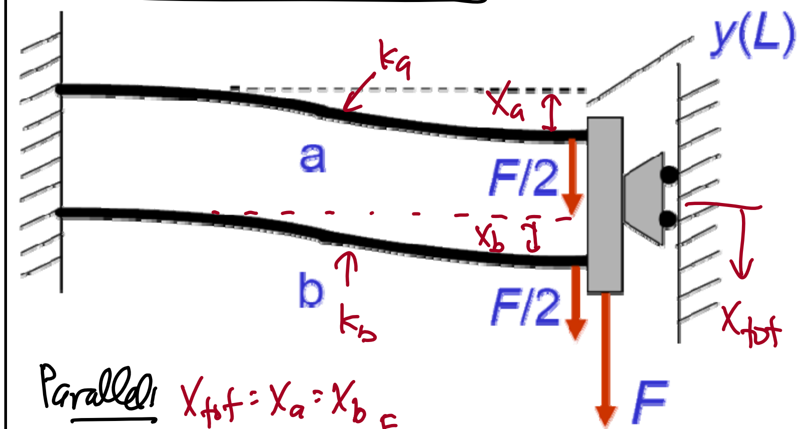
$$y(L) = \frac{F}{k} = 2y(L_c) = 2\left(\frac{F}{k_c}\right) = F\left(\frac{1}{k_c}+\frac{1}{k_c}\right)$$

stiffness of the whole thing

$$\boxed{\frac{1}{k} = \frac{1}{k_c}+\frac{1}{k_c}} \rightarrow \boxed{k = k_c \| k_c}$$

Definition for "||": $$A\|B = \frac{1}{\frac{1}{A}+\frac{1}{B}} = \frac{AB}{A+B}$$

Parallel Combination of Springs



Parallel: $x_{tot} = x_a = x_b$

$$y(L) = \frac{F}{k} = \frac{F_a}{k_a} = \frac{F_b}{k_b} = \left(\frac{F}{2}\right)\left(\frac{1}{k_a}\right), \quad F_a \to \frac{F}{2},\ F_b \to \frac{F}{2}$$

of the whole thing

$$\boxed{k = 2k_a}$$

In general: $\boxed{k_{tot} = k_a + k_b}$

For EE's: springs combine like capacitors

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