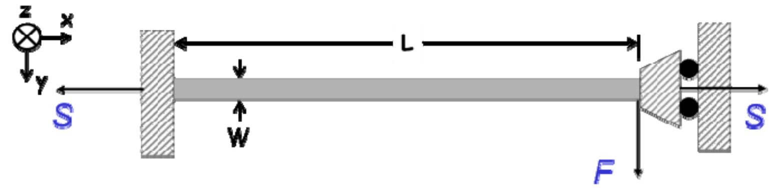
Lecture 17: Beam Combos II

- Announcements:
- No lecture this coming Thursday
  - We did the make-up for this last Friday
- Unbelievably, both recordings of the Make-Up lecture last Friday didn't work
  - So again, we only have audio
  - But there is the Lecture 16 from Fall 2010, which is very similar
  - So the audio is on the web and there is a link to the YouTube video of the Fall 2010 Lecture 16 that you can view if you missed Friday's lecture
- Midterm is nearing: Thursday, Oct. 27
  - I passed out materials associated with the midterm, including and information sheet and old exams
- HW#4 due today
- HW#5 online (soon, if not already)
- --
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- --
- Last Time:
- Going through effect of tension on the beam equation

Tensioned Spring (Non-Ideality)

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L



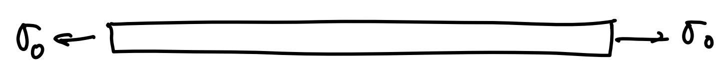
Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4 y}{dx^4} - S\frac{d^2 y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

Axial Load

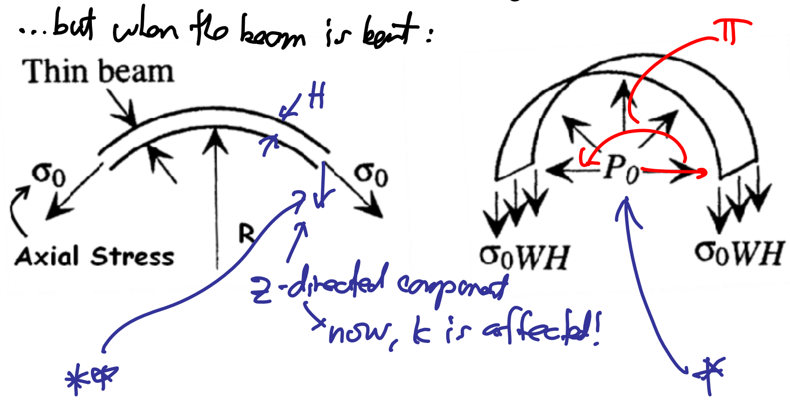
Heuristic Derivation for the Euler Beam Equation

Consider first a straight beam under an axial stress:



⇒ no effect on z-directed stiffness when the beam is straight

...but when the beam is bent:



Copyright © 2011 Regents of the University of California

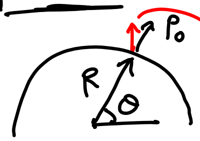
* Upward pressure P_o to counteract the downward force from ** to keep everything in static equilibrium

For ease of analysis:

Assume the beam is bent to an angle π

Downward vertical force: $2\sigma_o WH$

Upward Force due to P_o:



$P_y(\theta) = P_o \sin\theta$

$$F_u = \int_0^{\pi} (P_o \sin\theta)\, W (R\,d\theta)$$
$$= -P_o W R \cos\theta \Big|_0^{\pi}$$
$$= 2RWP_o$$

[Equilibrium] $\Rightarrow 2RWP_o = 2\sigma_o WH \rightarrow P_o = \dfrac{\sigma_o H}{R}$

$\left[q_o = \dfrac{\text{beam load}}{\text{unit length}} = P_o W,\ \dfrac{1}{R} = \dfrac{d^2 w}{dx^2} \right] \Rightarrow$ beam displacement

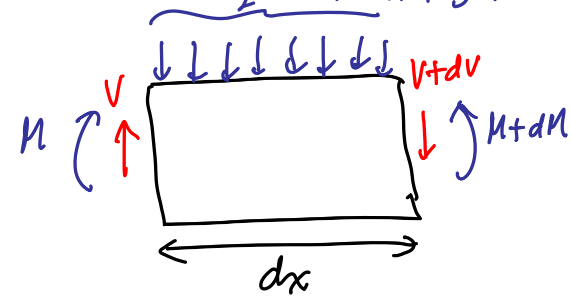
$q_o = \sigma_o WH \dfrac{d^2 w}{dx^2}$ ← generalizes to the case of smaller displacements & angles

Using the differential beam bending Equation

$\dfrac{d^2 w}{dx^2} = -\dfrac{M}{EI} \longrightarrow \dfrac{d^4 w}{dx^4} = \dfrac{q}{EI}$ ← $\dfrac{\text{load}}{\text{unit length}}$

† ??? *

* Relationships Between Forces on a Fully Loaded Differential Beam Element



[Total Static Equilibrium] → total force = 0

F_T = total force = $q\,dx + (\cancel{V} + dV) - \cancel{V} = 0$

$$\therefore \boxed{\frac{dV}{dx} = -q} \qquad (1)$$

→ also, total moment w/r to left hand edge = 0

$$M_T = (\cancel{M} + dM) - \cancel{M} - (V + \cancel{dV})dx - \cancel{\tfrac{1}{2} q\,dx^2} = 0$$

[neglect products of differentials] $\int_0^{dx} (q\,du)u = \frac{1}{2} q\,dx^2$

$$dM - V\,dx = 0 \rightarrow \boxed{\frac{dM}{dx} = V} \qquad (2)$$

Using (1) & (2):

$$\left[\frac{d^2 M}{dx^2} = \frac{dV}{dx} = -q \right] \quad †$$

← extend load

$$EI \frac{d^4 w}{dx^4} = q + q_o \leftarrow \text{equiv. load f/ axial stress}$$

Copyright © 2011 Regents of the University of California

$\left[q_0 = \sigma_0 W t \frac{d^2 w}{dx^2}\right] \rightarrow$

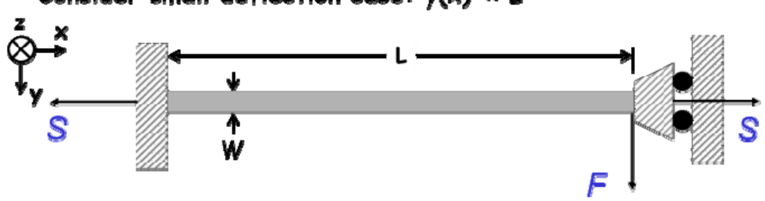
$$EI\frac{d^4 w}{dx^4} - \underbrace{(\sigma_0 W t)}_{\text{tension in the beam} = S}\frac{d^2 w}{dx^2} = q$$

S ← a force

Euler Beam Equation

Clamped-Guided Beam Under Axial Load

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$



Governing differential equation: (Euler Beam Equation)

$$EI_z\frac{d^4 y}{dx^4} - \underbrace{S}_{\text{Axial Load}}\frac{d^2 y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

- Can solve the ODE using standard methods
  - Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)
  - For solution to the clamped-guided case: see S. Timoshenko, Strength of Materials II: Advanced Theory and Problems, McGraw-Hill, New York, 3rd Ed., 1955
- Result from Timoshenko:

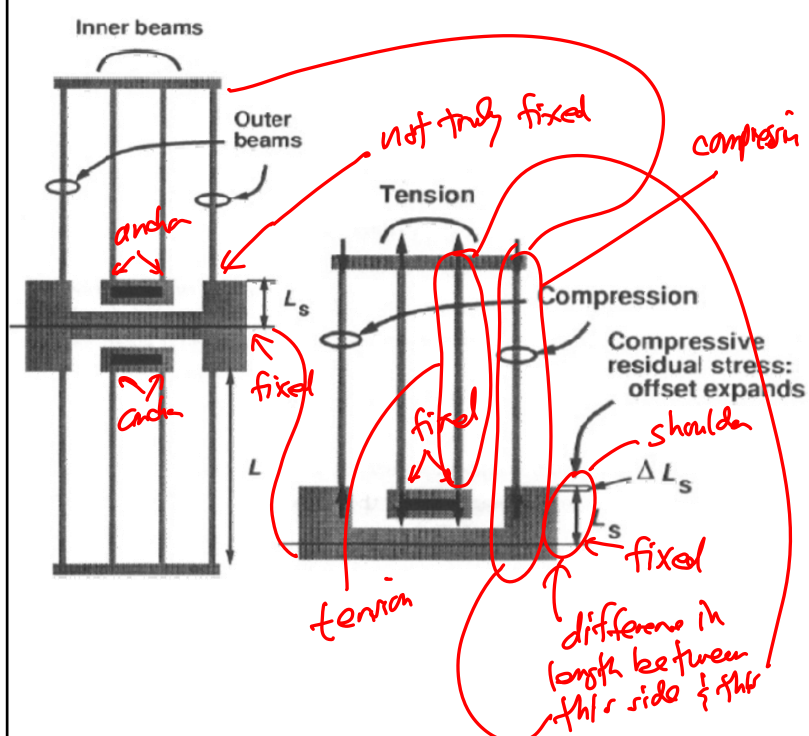
$S > 0$ (tension) $\quad k^{-1} = \frac{pL - 2\tanh(pL/2)}{p|S|} = \frac{y(x=L)}{F}$

$S < 0$ (compression)

$$k^{-1} = \frac{-pL + 2\tan(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

$|S|$ ← force

where $p = \sqrt{\frac{|S|}{EI_z}}$



Copyright © 2011 Regents of the University of California

① If polySi strain is ϵ_r, then shuttle expands by $\Delta L_s = \epsilon_r L_s$

② This then applies a load to the beams, $\Delta L = \Delta L_s$

③ Beam Stress:

$$\epsilon_b = \frac{\Delta L}{2L} = \frac{\Delta L_s}{2L} = \pm \epsilon_r \frac{L_s}{2L}$$

Stress Force: $S = \pm E\epsilon_r \left(\frac{L_s}{2L}\right) Wh$ (axial tension)

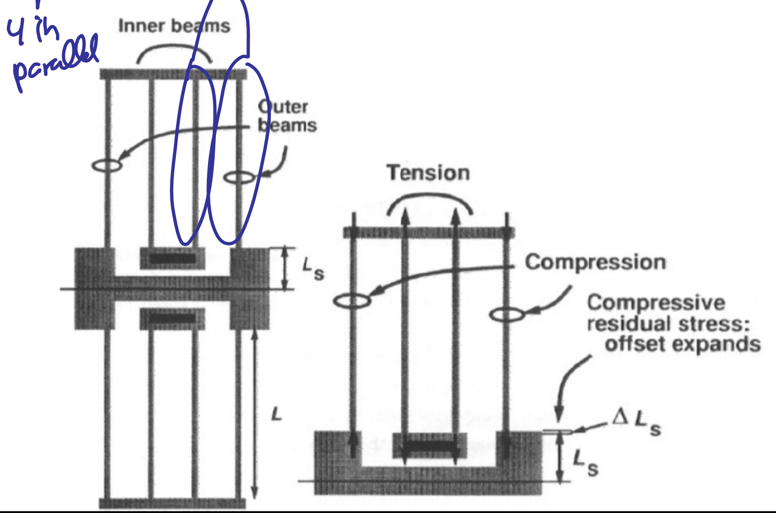
④ Spring Constants:

series combination of beams

$$k = 4(k_{com}^{-1} + k_{ten}^{-1})^{-1}$$

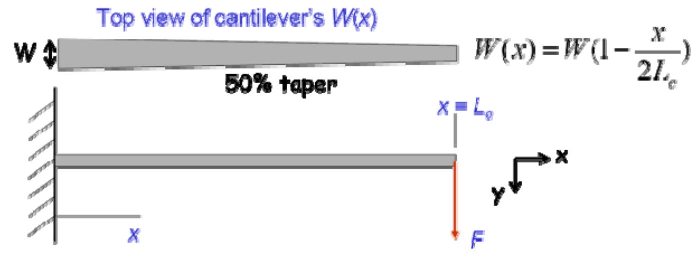
$$k = 4\left[\frac{-pL + 2\tan(pL/2)}{pS} + \frac{pL - 2\tanh(pL/2)}{p|S|}\right]^{-1}$$

4 in parallel



More General Geometries

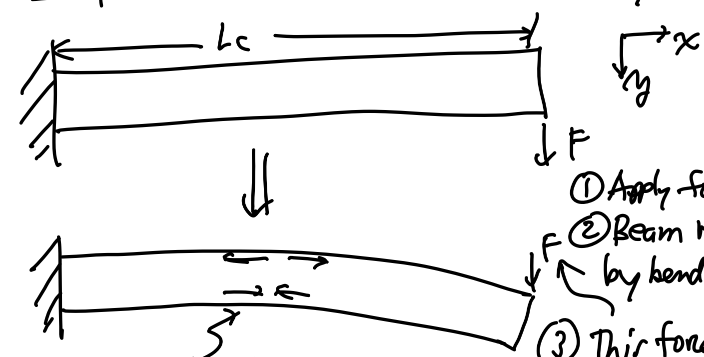
- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- **Example**: tapered cantilever beam
- **Objective**: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)



Look @ the "Principle of Work Slide" in Module 9.

More Visual Description

Same problem as before: Take a beam, apply a force.



① Apply force.
② Beam responds by bending.
③ This force has done work:
$W = F \cdot y(L_c)$
④ Strain generated
→ So the beam has received an influx of stored energy
→ magnitude of " " determined by shape

Copyright © 2011 Regents of the University of California

(5) Then

U = Stored Energy − Work Done $\rightarrow 0$

when we choose the
right shape.

This is how we get the
beam's response to F.

Copyright © 2011 Regents of the University of California