

Shear Strain Energy

$$W_{shear} = \frac{3(EI_z)^2 L}{4GWh} \int_0^L \left(\frac{d^3 y}{dx^3} \right)^2 dx$$

Shear Modulus

- See W.C. Albert, "Vibrating Quartz Crystal Beam Accelerometer," Proc. ISA Int. Instrumentation Symp., May 1982, pp. 33-44

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Applying the Principle of Virtual Work

- Basic Procedure:**
 - Guess the form of the beam deflection under the applied loads
 - Vary the parameters in the beam deflection function in order to minimize:

$$U = \underbrace{\sum_j W_j}_{\text{Sum strain energies}} - \sum_i F_i u_i$$

Assumes point load

Displacement at point load
- Find minima by simply setting derivatives to zero

- See Senturia, pg. 244, for a general expression with distributed surface loads and body forces

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Example: Tapered Cantilever Beam

- Objective:** Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width $W(x)$

Top view of cantilever's $W(x)$

$W(x) = W \left(1 - \frac{x}{2L_c} \right)$

50% taper

Adjustable parameters: minimize U

$y(x) = c_2 x^2 + c_3 x^3$

- Start by guessing the solution
 - It should satisfy the boundary conditions
 - The strain energy integrals shouldn't be too tedious
 - This might not matter much these days, though, since one could just use matlab or mathematica

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Strain Energy And Work By F

$$U = W_{bend} - F \cdot y(L_c)$$

$$W_{bend} = \frac{1}{2} E \int_0^{L_c} I_z(x) \left(\frac{d^2 y}{dx^2} \right)^2 dx \quad \text{(Bending Energy)}$$

$$I_z(x) = \frac{W(x)h^3}{12} \quad \frac{d^2 y}{dx^2} = 2c_2 + 6c_3 x$$

(Using our guess)

$$W(x) = W \left(1 - \frac{x}{2L_c} \right)$$

Tip Deflection

$$= \frac{1}{24} E W h^3 \int_0^{L_c} \left(1 - \frac{x}{2L_c} \right) (2c_2 + 6c_3 x)^2 dx - F (c_2 L_c^2 + c_3 L_c^3)$$

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Find c_2 and c_3 That Minimize U

- Minimize $U \rightarrow$ basically, find the c_2 and c_3 that brings U closest to zero (which is what it would be if we had guessed correctly)
- The c_2 and c_3 that minimize U are the ones for which the partial derivatives of U with respect to them are zero:

$$\frac{\partial U}{\partial c_2} = 0 \quad \frac{\partial U}{\partial c_3} = 0$$

- Proceed:
 - First, evaluate the integral to get an expression for U :

$$U = EWh^3 \left\{ \frac{5c_3^2}{16} L_e^3 + \frac{c_2 c_3}{3} L_e^2 + \frac{c_2^2}{8} L_e \right\} - F(c_2 L_e^2 + c_3 L_e^3)$$

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Minimize U (cont)

- Evaluate the derivatives and set to zero:

$$\frac{\partial U}{\partial c_2} = 0 = \left(\frac{EWh^3}{3} c_3 - F \right) L_e^2 + \left(\frac{EWh^3}{4} c_2 \right) L_e$$

$$\frac{\partial U}{\partial c_3} = 0 = \left(\frac{5}{8} EWh^3 c_3 - F \right) L_e^3 + \left(\frac{EWh^3}{3} c_2 \right) L_e^2$$

- Solve the simultaneous equations to get c_2 and c_3 :

$$c_2 = \left(\frac{84}{13} \right) \frac{FL_e}{EWh^3} \quad c_3 = - \left(\frac{24}{13} \right) \frac{F}{EWh^3}$$

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The Virtual Work-Derived Solution

- And the solution:

$$y(x) = \left(\frac{24F}{13EWh^3} \right) \left(\left(\frac{7}{2} \right) L_e - x \right) x^2$$

- Solve for tip deflection and obtain the spring constant:

$$y(L_e) = \left(\frac{24F}{13EWh^3} \right) \left(\frac{5}{2} \right) L_e^3 \quad k_e = F / y(L_e) = \left(\frac{13EWh^3}{60L_e^3} \right)$$

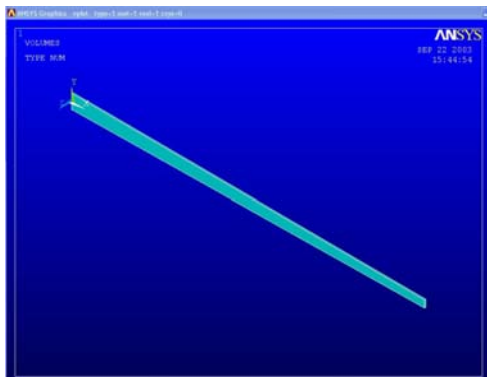
- Compare with previous solution for constant-width cantilever beam (using Euler theory):

$$y(L_e) = \left(\frac{4F}{EWh^3} \right) L_e^3 \rightarrow \text{13\% smaller than tapered-width case}$$

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Comparison With Finite Element Simulation

- Below: ANSYS finite element model with
 - $L = 500 \mu\text{m}$ $W_{\text{base}} = 20 \mu\text{m}$ $E = 170 \text{ GPa}$
 - $h = 2 \mu\text{m}$ $W_{\text{tip}} = 10 \mu\text{m}$



- Result: (from static analysis)
 - $k = 0.471 \mu\text{N/m}$
- This matches the result from energy minimization to 3 significant figures

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 **Need a Better Approximation?**

- Add more terms to the polynomial
- Add other strain energy terms:
 - ↳ Shear: more significant as the beam gets shorter
 - ↳ Axial: more significant as deflections become larger
- Both of the above remedies make the math more complex, so encourage the use of math software, such as Mathematica, Matlab, or Maple
- Finite element analysis is really just energy minimization
- If this is the case, then why ever use energy minimization analytically (i.e., by hand)?
 - ↳ Analytical expressions, even approximate ones, give insight into parameter dependencies that FEA cannot
 - ↳ Can compare the importance of different terms
 - ↳ Should use in tandem with FEA for design

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