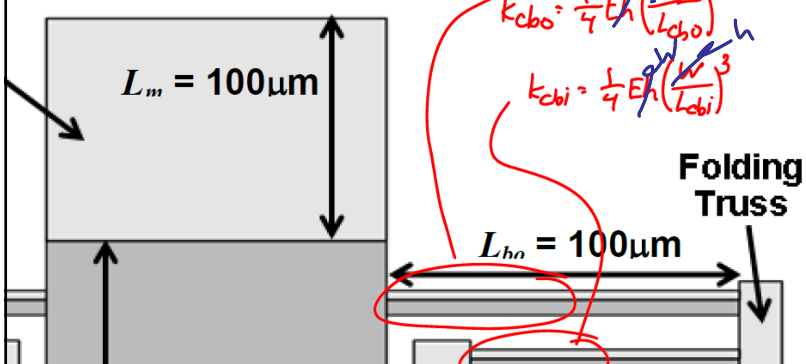
Lecture 20: Equivalent Circuits

- **Announcements**:
- HW#6 online yesterday
- Modules 10 and 11 online
- Project handed out and described last time
- Graded midterm passed out last time
- Slight copying error on exam solutions for problem 4(c):



$k_{cbo} = \frac{1}{4}Eh\left(\frac{W}{L_{cbo}}\right)^3$

$k_{cbi} = \frac{1}{4}Eh\left(\frac{W}{L_{cbi}}\right)^3$

$$k_p = 4\left(\frac{k_{cbi}}{2} \,\|\, \frac{k_{cbo}}{2}\right) = 4\left(\frac{1}{8}Eh\left(\frac{W}{L_{cbi}}\right)^3 \,\|\, \frac{1}{8}Eh\left(\frac{W}{L_{cbo}}\right)^3\right)$$

$$= \frac{1}{2}EhW^3\left(\frac{1}{L_{cbi}^3} \,\|\, \frac{1}{L_{cbo}^3}\right) = \frac{1}{2}EhW^3\left(\frac{1}{L_{cbi}^3 + L_{cbo}^3}\right)$$

Thus, need:

$$(g\cdot s)^2 = \frac{8(L_m\cdot s)^2\gamma_{la}\cos\theta_c}{\frac{1}{2}E(hs)(Ws)^3}\left((sL_{cbi})^3 + (sL_{cbo})^3\right)$$

$$g^2\cdot s^2 = \frac{8L_m^2\gamma_{la}\cos\theta_c}{\frac{1}{2}EhW^3}\left(L_{cbi}^3 + L_{cbo}^3\right)\cdot\underbrace{\frac{s^2}{s^4}\cdot s^3}_{s}$$

$\therefore$ need $$s = \frac{16L_m^2\gamma_{la}\cos\theta_c}{g^2EhW^3}\left(L_{cbi}^3 + L_{cbo}^3\right)$$

$$s = \frac{16(100\mu)^2(72.75\times10^{-3})\cos(85^\circ)}{(5\mu)^2(150G)(5\mu)(2\mu)^3}\left((75\mu)^3 + (100\mu)^3\right)$$

$\rightarrow s = 5.2 \rightarrow 0.19$

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- Reading: Senturia, Chpt. 5
- Lecture Topics:
  - Lumped Mechanical Equivalent Circuits
  - Electromechanical Analogies
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
  - Parallel-Plate Capacitive Transducers
- --
- **Last Time**:
- Derived the following for the resonance frequency of a folded beam resonator:

$$\omega_0 = \left[\frac{k_x}{M_{eq}}\right]^{1/2}$$

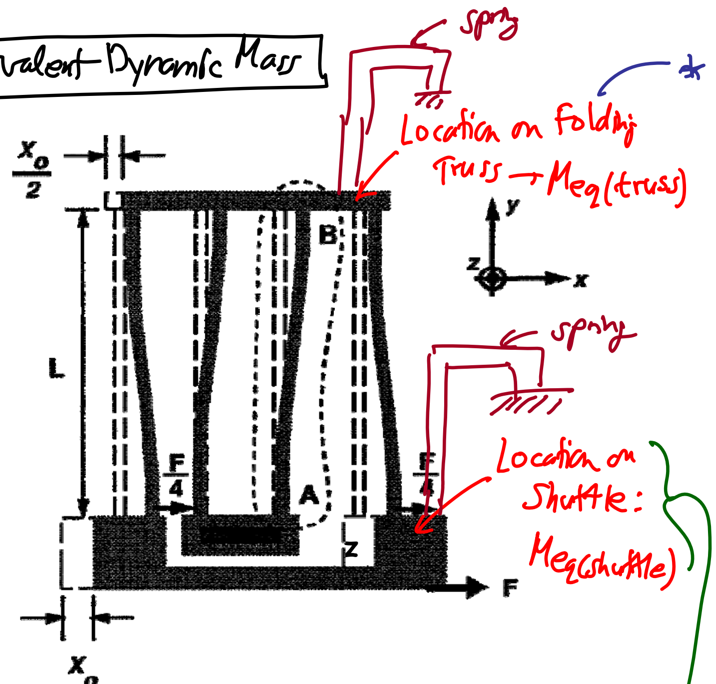
where $M_{eq} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b$

(k_c; dynamic mass)

(Resonance Freq. of a Folded-Beam Suspended Shuttle)

- Go through Module 10 slides 21-31 on your own

Equivalent Dynamic Mass



Equivalent Mass:

$$\text{Equiv. Mass} = M_{eq,x} = \frac{KE_{max}}{\frac{1}{2}v_x^2} = \frac{\frac{1}{2}\rho A \int_0^l v^2(x)dx}{\frac{1}{2}v_x^2}$$

(v_x ← velocity @ location x)

$$M_{eq(shuttle)} = \frac{KE_{max}}{\frac{1}{2}V_{shuttle}^2} = \frac{\omega_0^2 X_0^2 (\frac{1}{2})[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b]}{\frac{1}{2}\omega_0^2 x_0^2}$$

$$M_{eq(shuttle)} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b$$

static masses = ρ(Volume)

* —
$$M_{eq(truss)} = \frac{\omega_0^2 x_0^2 (\frac{1}{2})[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b]}{\frac{1}{2}(\frac{1}{4})\omega_0^2 x_0^2}$$

$$M_{eq(truss)} = 4[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b]$$

↑ Equiv. Dynamic Mass

Equiv. Dynamic Stiffness

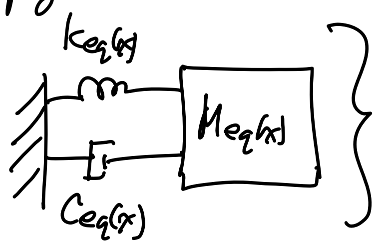
$$\omega_0 = \sqrt{\frac{K_{eq}(x)}{M_{eq}(x)}} \rightarrow K_{eq}(x) = \omega_0^2 M_{eq}(x)$$

⇒ large equiv. mass & large equiv. stiffness go hand-in-hand

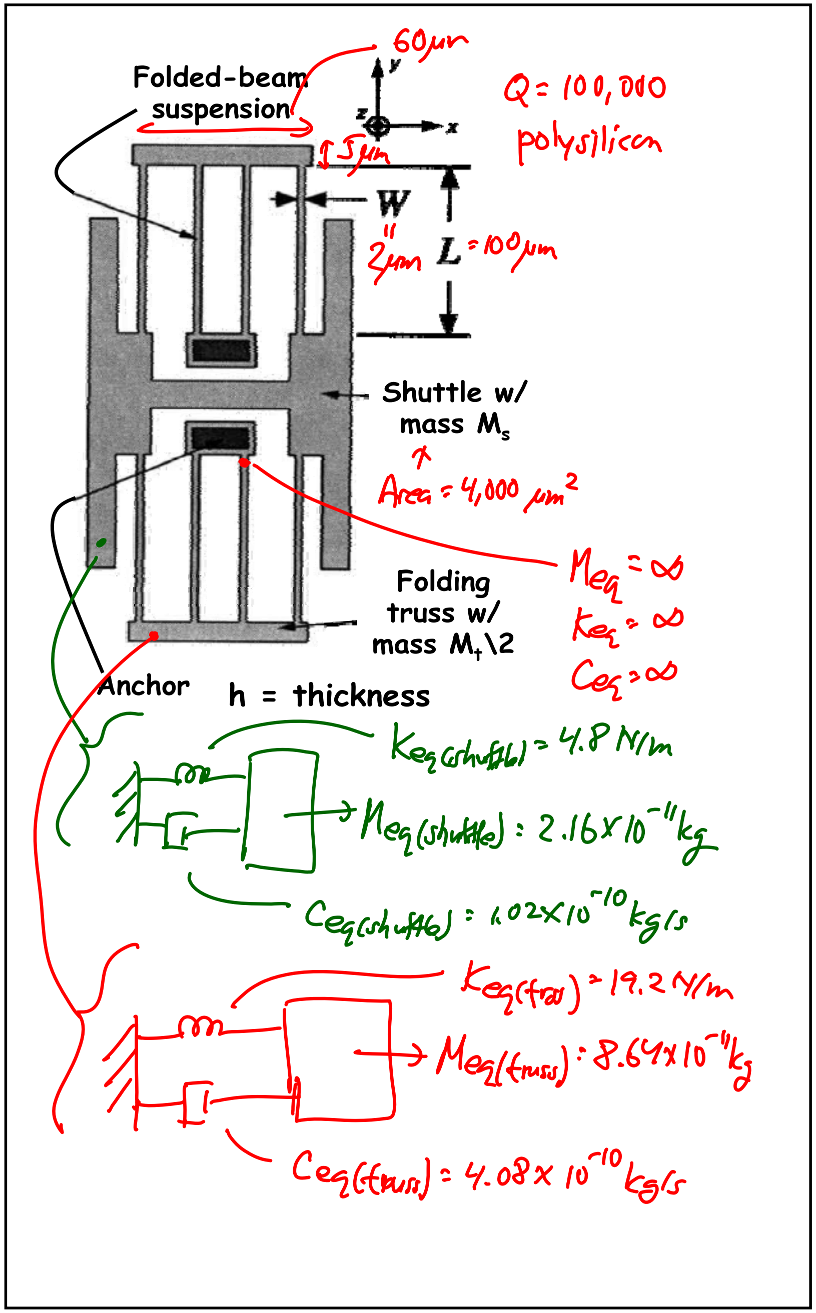
Equiv. Dynamic Damping

$$Q = \frac{\omega_0 M_{eq}(x)}{C_{eq}(x)} \rightarrow C_{eq}(x) = \frac{\omega_0 M_{eq}(x)}{Q} = \frac{\sqrt{K_{eq}(x) M_{eq}(x)}}{Q}$$

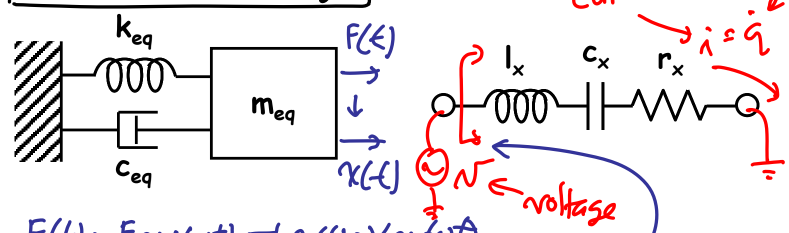
($M_{eq}(x) \sim L$, $C_{eq}(x) \sim R$; $C_{eq}(x)$ ← damping)



} specified @ a single location x



Electromechanical Analogies



$F(t) = F\cos(\omega t) \rightarrow x(t) = X\cos(\omega t)$ (off resonance)

Equation of Motion:

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = F(t)$$

⇒ using phasor concepts:

$$F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + c_{eq}\dot{x}$$

Impedance looking in

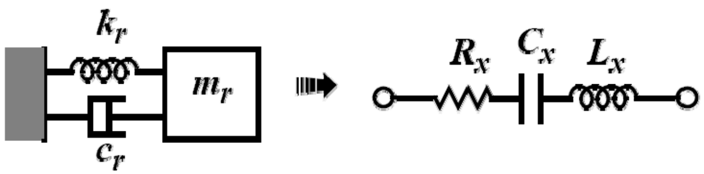
$$\frac{v}{i} = j\omega l_x + \frac{1}{j\omega C_x} + r_x$$

$$v = j\omega l_x i + \frac{(1/C_x)}{j\omega} i + r_x i$$

⇒ by analogy:

$F \rightarrow v$	$m_{eq} \rightarrow l_x$	$c_{eq} \rightarrow r_x$
$\dot{x} \rightarrow i$	$k_{eq} \rightarrow \frac{1}{C_x}$	

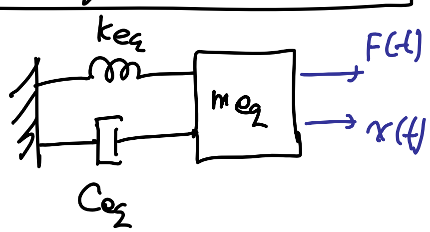
[Parameter Relationships in the Current Analogy]



- Mechanical-to-electrical correspondence in the current analogy:

Mechanical Variable	Electrical Variable
Damping, c	Resistance, R
Stiffness^{-1}, k^{-1}	Capacitance, C
Mass, m	Inductance, L
Force, f	Voltage, V
Velocity, v	Current, I

Lowpass Biquad Transfer Function



$$F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + c_{eq}\dot{x}$$

⇒ convert to full phasor form:

$$F = (j\omega)(j\omega X)\, m_{eq} + \frac{k_{eq}}{j\omega}(j\omega X) + c_{eq}(j\omega X)$$

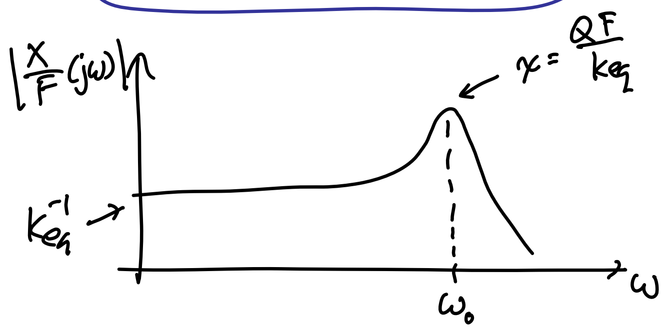
$$\frac{X}{F}(j\omega) = \frac{1}{k_{eq}}\left[-\omega^2\frac{m_{eq}}{k_{eq}} + 1 + j\frac{c_{eq}\omega}{k_{eq}}\right]^{-1} \text{ —— } *$$

$$\left[\frac{k_{eq}}{m_{eq}} = \omega_0^2,\quad Q = \frac{m_{eq}\omega_0}{c_{eq}} = \frac{k_{eq}}{\omega_0 c_{eq}} \rightarrow \frac{k_{eq}}{c_{eq}} = Q\omega_0\right]$$

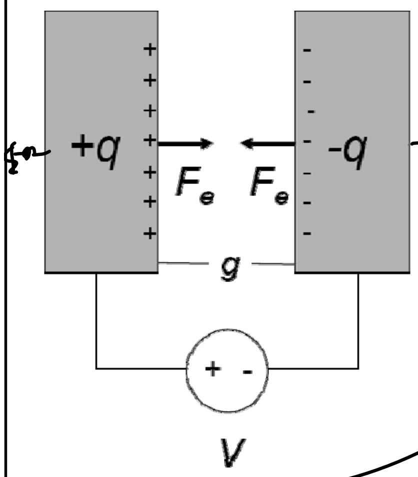
$*$ →

$$\frac{X}{F}(j\omega) = \frac{1}{k_{eq}}\left[-\left(\frac{\omega}{\omega_0}\right)^2 + 1 + j\frac{\omega}{Q\omega_0}\right]^{-1}$$

$$\frac{X}{F}(j\omega) = \frac{k_{eq}^{-1}}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + j\frac{\omega}{Q\omega_0}}$$



Basic Physics of Electrostatic Actuation



Goal: Determine gap spacing g as a function of input variable

Note: Assume the plates are supported elastically

1st: Determine the energy of the system.

2nd: Ask, What can I do to Δ the energy of the system?

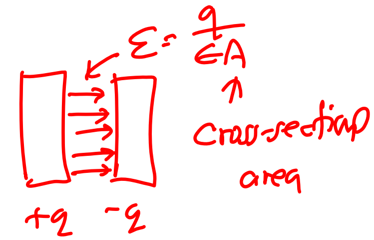
① change the charge q

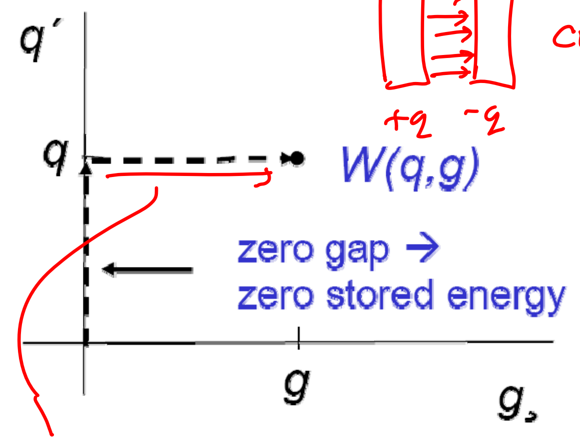
② change the separation g

$$\Delta W(q,g) = V\Delta q + F_e \Delta g$$

$$dW = V dq + F_e dg$$

Stored Energy





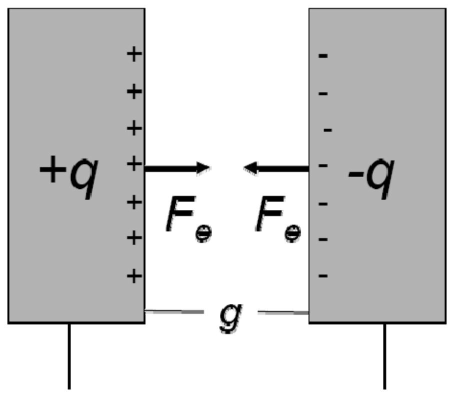
No change in charge: $dq = 0$

$$W = 0 + \int_0^g F_e \, dg'$$

$$F_e = \left(\frac{q}{2}\right)\mathcal{E} = \frac{1}{2}\frac{q^2}{\epsilon A} \quad \text{(independent of } g\text{)}$$

$$\therefore W = \int_0^g F_e \, dg' = F_e g' \Big|_0^g = F_e g$$

$$W(g) = \frac{1}{2}\frac{q^2}{\epsilon A} g$$



From $dW = V dq + F_e dg$

⇒ Force is given by:

$$F_e = \left.\frac{\partial W(q,g)}{\partial g}\right|_{q = \text{const.}} = \frac{\partial}{\partial g}\left(\frac{1}{2}\frac{q^2}{\epsilon A} g\right)$$

$$\therefore \boxed{F_e = \frac{1}{2}\frac{q^2}{\epsilon A}}$$ → indep. of gap spacing!