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# EE C245 - ME C218 Introduction to MEMS Design Fall 2011

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Lecture Module 10: Resonance Frequency

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## Lecture Outline


- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - ↗ Estimating Resonance Frequency
  - ↗ Lumped Mass-Spring Approximation
  - ↗ ADXL-50 Resonance Frequency
  - ↗ Distributed Mass & Stiffness
  - ↗ Folded-Beam Resonator

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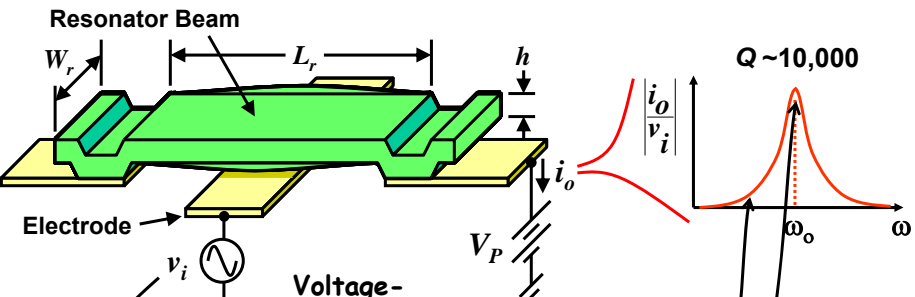


## Estimating Resonance Frequency

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## Clamped-Clamped Beam $\mu$ Resonator



Resonator Beam  
 $W_r$ ,  $L_r$ ,  $h$

Electrode  
 $v_i$

Sinusoidal Excitation

Voltage-to-Force Capacitive Transducer  
 $V_P$

Sinusoidal Forcing Function

$i_o$

$Q \sim 10,000$

$\omega_0$ ,  $\omega$

$$v_i = V_i \cos[\omega_o t] \rightarrow f_i = F_i \cos[\omega_o t]$$

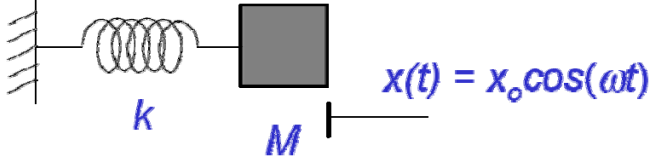
- $\omega \neq \omega_o$ : small amplitude
- $\omega = \omega_o$ : maximum amplitude  $\rightarrow$  beam reaches its maximum potential and kinetic energies

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**Estimating Resonance Frequency**

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- Assume simple harmonic motion:



$x(t) = x_o \cos(\omega t)$

- Potential Energy:

$$W(t) = \frac{1}{2} kx^2(t) = \frac{1}{2} kx_o^2 \cos^2(\omega t)$$

- Kinetic Energy:

$$K(t) = \frac{1}{2} M\dot{x}^2(t) = \frac{1}{2} Mx_o^2 \omega^2 \sin^2(\omega t)$$

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**Estimating Resonance Frequency (cont)**

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- Energy must be conserved:
  - Potential Energy + Kinetic Energy = Total Energy
  - Must be true at every point on the mechanical structure

Occurs at peak displacement      Occurs when the beam moves through zero displacement

Maximum Potential Energy      Maximum Kinetic Energy

$$W_{\max} = \frac{1}{2} kx_o^2 = K_{\max} = \frac{1}{2} M\omega^2 x_o^2$$

Stiffness      Displacement Amplitude      Mass      Radian Frequency

- Solving, we obtain for resonance frequency:

$$\omega = \sqrt{\frac{k}{M}}$$

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**Example: ADXL-50**

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- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam:  $L = 260 \mu\text{m}$ ,  $h = 2.3 \mu\text{m}$ ,  $W = 2 \mu\text{m}$

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**Lumped Spring-Mass Approximation**

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- Mass is dominated by the proof mass
  - 60% of mass from sense fingers
  - Mass =  $M = 162 \text{ ng}$  (nano-grams)
- Suspension: four tensioned beams
  - Include both bending and stretching terms [A.P. Pisano, BSAC Inertial Sensor Short Courses, 1995-1998]

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## ADXL-50 Suspension Model

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- Bending contribution:
 
$$k_b^{-1} = (1/k_c + 1/k_e) = 2 \left[ \frac{(L/2)^3}{3E(Wh^3/12)} \right] = \frac{L^3}{EWh^3} = 4.2 \mu\text{m} / \mu\text{N}$$
- Stretching contribution:
 
$$k_{st}^{-1} = L/S = \frac{L}{\sigma_y Wh} = 1.14 \mu\text{m} / \mu\text{N}$$

$F_y = S \sin \theta \approx S(x/L) = \underbrace{\left(\frac{S}{L}\right)}_{k_{st}} x$
- Total spring constant: *add bending to stretching*  
*(since they are in parallel)*

$$k = 4(k_b + k_{st}) = 4(0.24 + 0.88) = 4.5 \mu\text{N} / \mu\text{m}$$


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## ADXL-50 Resonance Frequency

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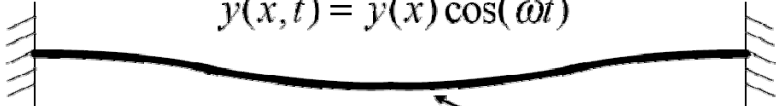
- Using a lumped mass-spring approximation:
 
$$f = \frac{1}{2\pi} \sqrt{\frac{k}{M}} = \frac{1}{2\pi} \sqrt{\frac{4.48 \text{ N/m}}{162 \times 10^{-12} \text{ kg}}} = 26.5 \text{ kHz}$$
- On the ADXL-50 Data Sheet:  $f_0 = 24 \text{ kHz}$ 
  - ↗ Why the 10% difference?
  - ↗ Well, it's approximate ... plus ...
  - ↗ Above analysis does not include the frequency-pulling effect of the DC bias voltage across the plate sense fingers and stationary sense fingers ... something we'll cover later on ...

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## Distributed Mechanical Structures

- Vibrating structure displacement function:
 

$$y(x, t) = \hat{y}(x) \cos(\omega t)$$



Maximum displacement function  
 (i.e., mode shape function)  
 Seen when velocity  $\dot{y}(x, t) = 0$
- Procedure for determining resonance frequency:
  - ↪ Use the static displacement of the structure as a trial function and find the strain energy  $W_{max}$  at the point of maximum displacement (e.g., when  $t=0, \pi/\omega, \dots$ )
  - ↪ Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - ↪ Equate energies and solve for frequency

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## Maximum Kinetic Energy

- Displacement:  $y(x, t) = \hat{y}(x) \cos[\omega t]$
- Velocity:  $v(x, t) = \frac{\partial y(x, t)}{\partial t} = -\omega \hat{y}(x) \sin[\omega t]$
- At times  $t = \pi/(2\omega), 3\pi/(2\omega), \dots$



$y(x, t) = 0$

Velocity topographical mapping

- ↪ The displacement of the structure is  $y(x, t) = 0$
- ↪ The velocity is maximum and all of the energy in the structure is kinetic (since  $W=0$ ):
 

$$v(x, (2n + 1) \pi / (2\omega)) = -\omega \hat{y}(x)$$

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## Maximum Kinetic Energy (cont)

- At times  $t = \pi/(2\omega), 3\pi/(2\omega), \dots$

**Velocity:**  $v(x, (2n+1)\pi/(2\omega)) = -\omega \hat{y}(x)$

$$dK = \frac{1}{2} \cdot dm \cdot [v(x, t)]^2$$

$$dm = \rho(Wh \cdot dx)$$

- Maximum kinetic energy:

$$K_{\max} = \int_0^L \frac{1}{2} \rho Wh dx v^2(x, t) = \int_0^L \frac{1}{2} \rho Wh \omega^2 \hat{y}^2(x) dx$$

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## The Raleigh-Ritz Method

- Equate the maximum potential and maximum kinetic energies:

$$K_{\max} = \int_0^L \frac{1}{2} \rho Wh \omega^2 \hat{y}^2(x) dx = \mathcal{W}_{\max}$$

- Rearranging yields for resonance frequency:

$$\omega = \sqrt{\frac{\mathcal{W}_{\max}}{\int_0^L \frac{1}{2} \rho Wh \hat{y}^2(x) dx}}$$

$\omega$  = resonance frequency  
 $\mathcal{W}_{\max}$  = maximum potential energy  
 $\rho$  = density of the structural material  
 $W$  = beam width  
 $h$  = beam thickness  
 $\hat{y}(x)$  = resonance mode shape

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**Example: Folded-Beam Resonator**

• Derive an expression for the resonance frequency of the folded-beam structure at left.

Use Rayleigh-Ritz method.

$$KE_{max} = PE_{max}$$

Kinetic Energy:

$$KE_{max} = \underbrace{KE_S}_{\text{shuttle}} + \underbrace{KE_t}_{\text{truss}} + \underbrace{KE_b}_{\text{beams}}$$

$$= \frac{1}{2} N_s^2 M_s + \frac{1}{2} N_t^2 M_t + \frac{1}{2} \int N_b^2 dM_b$$

mass of both trusses → Must integrate since the beam velocity is a function of location  $y$ !

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**Get Kinetic Energies**

Velocity of the shuttle:  $N_s = \omega_0 \Delta_0$   
 Resonance Freq. → Maximum Displacement Amplitude

$$\therefore KE_S = \frac{1}{2} N_s^2 M_s = \frac{1}{2} \omega_0^2 \Delta_0^2 M_s$$

Velocity of the truss:  $N_t = \frac{1}{2} N_s = \frac{1}{2} \omega_0 \Delta_0$

$$\therefore KE_t = \frac{1}{2} \left( \frac{1}{2} \omega_0 \Delta_0 \right)^2 M_t = \frac{1}{8} \omega_0^2 \Delta_0^2 M_t$$

Velocity of the beam segments:  
 ⇒ assume the mode shape is the same as the static displacement shape  
 ⇒ For segment AB:

$$\hat{x}(y) = \frac{F_x}{48 E I_z} (3Ly^2 - 2y^3) \quad 0 \leq y \leq L \quad (1)$$

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**Folded-Beam Suspension**

**Comb-Driven Folded Beam Actuator**

$$\hat{x}(y) = \frac{F_x}{48EI_z} (3Ly^2 - 2y^3) \quad 0 \leq y \leq L$$

Case:  $y=0 \quad \hat{x}(y=0) = 0 \quad \checkmark$

Case:  $y=L \quad \hat{x}(y=L) = \frac{F_x}{48EI_z} L^3 \rightarrow k = \frac{(F_x/L)}{x} = \frac{12EI_z}{L^3} = \frac{k_c}{2} \quad \checkmark$

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**Get Kinetic Energies (cont)**

At  $y=L: x(L) = \frac{X_0}{2} = \frac{F_x L^3}{48EI_z}$

Substituting into (1):

$$\hat{x}(y) = \frac{X_0}{2} \left[ 3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3 \right]$$

Which yields the velocity:

$$v_b(y)_{[AB]} = \frac{X_0}{2} \left[ 3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3 \right] \omega_0$$

Plugging into the expression for  $KE_b$ :

$$KE_{[AB]} = \frac{1}{2} \int_0^L \frac{X_0^2 \omega_0^2}{4} \left[ 3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3 \right]^2 dM_{[AB]}$$

Static mass of beam [AB] =  $\frac{X_0^2 \omega_0^2 M_{[AB]}}{8L} \int_0^L \left[ 3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3 \right]^2 dy$

$$KE_{[AB]} = \frac{13}{280} X_0^2 \omega_0^2 M_{[AB]}$$

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### Get Kinetic Energies (cont)

For segment CD:

$$v_b(y)|_{CD} = X_0 \left[ 1 - \frac{3}{2} \left( \frac{y}{L} \right)^2 + \left( \frac{y}{L} \right)^3 \right] \omega_0$$

Thus:

$$KE_{CD} = \frac{X_0^2 \omega_0^2 M_{CD}}{2L} \int_0^L \left[ 1 - \frac{3}{2} \left( \frac{y}{L} \right)^2 + \left( \frac{y}{L} \right)^3 \right]^2 dy$$

$$KE_{CD} = \frac{83}{280} X_0^2 \omega_0^2 M_{CD}$$

Static mass of beam [CD]

Let  $M_b \hat{=}$  total mass of the 8 beams.

Then:  $M_{AB} = M_{CD} = \frac{1}{8} M_b$

Thus:

$$KE_b = 4 KE_{AB} + 4 KE_{CD} = \frac{6}{35} X_0^2 \omega_0^2 M_b$$

and

$$KE_{max} = X_0^2 \omega_0^2 \left[ \frac{1}{2} M_s + \frac{1}{8} M_t + \frac{6}{35} M_b \right]$$

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### Get Potential Energy & Frequency

$PE_{max}$  is simply the work done to achieve maximum deflection:

$$PE_{max} = \frac{1}{2} k_x X_0^2 = k_c X_0^2$$

Thus, using Rayleigh-Ritz:

$$KE_{max} = PE_{max}$$

$$X_0^2 \omega_0^2 \left[ \frac{1}{2} M_s + \frac{1}{8} M_t + \frac{6}{35} M_b \right] = \frac{1}{2} k_x X_0^2$$

$$\omega_0 = \left[ \frac{k_x}{M_{eq}} \right]^{1/2} = k_c$$

where  $M_{eq} = M_s + \frac{1}{4} M_t + \frac{12}{35} M_b$

(Resonance Frequency of a Folded-Beam Suspended Shuttle)

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## Brute Force Methods for Resonance Frequency Determination

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## Basic Concept: Scaling Guitar Strings

**Guitar String**

Vib. Amplitude

Low Q

High Q

110 Hz

Freq.

Vibrating "A" String (110 Hz)

**Freq. Equation:**

$$f_o = \frac{1}{2\pi} \sqrt{\frac{k_r}{m_r}}$$

Stiffness

Mass

Guitar

**μMechanical Resonator**

Metallized Electrode

$W_r$

$L_r$

Anchor

Polysilicon Clamped-Clamped Beam

$h_r$

[Bannon 1996]

Transmission [dB]

$f_o = 8.5\text{MHz}$

$Q_{vac} = 8,000$

$Q_{air} \sim 50$

**Performance:**

$L_r = 40.8\mu\text{m}$

$m_r \sim 10^{-13}\text{ kg}$

$W_r = 8\mu\text{m}, h_r = 2\mu\text{m}$

$d = 1000\text{\AA}, V_p = 5\text{V}$

Press. = 70mTorr

Frequency [MHz]

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### Anchor Losses

$Q = 300$  at 70MHz

**Problem:** direct anchoring to the substrate  $\Rightarrow$  anchor radiation into the substrate  $\Rightarrow$  lower  $Q$

$Q = 15,000$  at 92MHz

**Solution:** support at motionless nodal points  $\Rightarrow$  isolate resonator from anchors  $\Rightarrow$  less energy loss  $\Rightarrow$  higher  $Q$

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### 92 MHz Free-Free Beam $\mu$ Resonator

• Free-free beam  $\mu$ mechanical resonator with non-intrusive supports  $\Rightarrow$  reduce anchor dissipation  $\Rightarrow$  higher  $Q$

**Design/Performance:**  
 $L_f = 13.1\mu\text{m}$ ,  $W_f = 6\mu\text{m}$   
 $h = 2\mu\text{m}$ ,  $d = 1000\text{\AA}$   
 $V_p = 28-76\text{V}$ ,  $W_s = 2.8\mu\text{m}$   
 $f_0 = 92.25\text{MHz}$   
 $Q \sim 7,450$  @ 10mTorr

[Wang, Yu, Nguyen 1998]      EE C245: Introduction to MEMS Design      LecM 10      C. Nguyen      11/4/08      24

**Higher Order Modes for Higher Freq.**

**2nd Mode Free-Free Beam**

**3rd Mode Free Free Beam**

Distinct Mode Shapes

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**Flexural-Mode Beam Wave Equation**

z  
u ← Transverse Displacement  
W = width  
x  
L  
h

$\rho A dx \frac{\partial^2 u}{\partial t^2} = ma$  ← inertial action

Free Body Diagram

internal actions (shear force & moments)

• Derive the wave equation for transverse vibration:  
 Dynamic Equilibrium Condition for forces in the y-direction:  $M + \frac{\partial M}{\partial x} dx - F - (F + \frac{\partial F}{\partial x} dx) - \rho A dx \frac{\partial^2 u}{\partial t^2} = 0$  neglect the  $\frac{\partial F}{\partial x} dx$  term

and the moment equilibrium condition:  $-F dx + \frac{\partial M}{\partial x} dx \approx 0$  (2)

Combining (1) & (2):

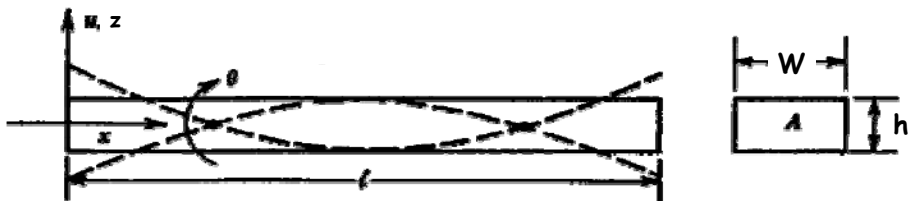
$$\frac{\partial^2 M}{\partial x^2} dx = -\rho A dx \frac{\partial^2 u}{\partial t^2} \rightarrow \frac{\partial^2}{\partial x^2} \left( -EI \frac{\partial^2 u}{\partial x^2} \right) = -\rho A \frac{\partial^2 u}{\partial t^2} \rightarrow \frac{\partial^2 u}{\partial t^2} = \left( \frac{EI}{\rho A} \right) \frac{\partial^4 u}{\partial x^4}$$

$\left[ \frac{\partial^2 u}{\partial x^2} = -\frac{M}{EI} \rightarrow M = -EI \frac{\partial^2 u}{\partial x^2} \right]$

$I_y = \frac{Wh^3}{12}$

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**Example: Free-Free Beam**



- Determine the resonance frequency of the beam
- Specify the lumped parameter mechanical equivalent circuit
- Transform to a lumped parameter electrical equivalent circuit
- Start with the flexural-mode beam equation:

$$\frac{\partial^2 u}{\partial t^2} = \left( \frac{EI}{\rho A} \right) \frac{\partial^4 u}{\partial x^4}$$

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**Free-Free Beam Frequency**

- Substitute  $u = u_1 e^{j\omega t}$  into the wave equation:

$$\frac{\partial^4 u}{\partial x^4} = \left( \omega^2 \frac{\rho A}{EI} \right) u \quad (1)$$

- This is a 4<sup>th</sup> order differential equation with solution:

$$u(x) = A \cosh kx + B \sinh kx + C \cos kx + D \sin kx \quad (2)$$

*Give the mode shape during resonance vibration.*

- Boundary Conditions:

At $x = 0$	At $x = l$	
$\frac{\partial^2 u}{\partial x^2} = 0$	$\frac{\partial^2 u}{\partial x^2} = 0$	$M = 0$ (Bending moment)
$\frac{\partial^3 u}{\partial x^3} = 0$	$\frac{\partial^3 u}{\partial x^3} = 0$	$\frac{\partial M}{\partial x} = 0$ (Shearing force)

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**Free-Free Beam Frequency (cont)**

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- Applying B.C.'s, get  $A=C$  and  $B=D$ , and
 
$$\begin{bmatrix} (\cosh kl - \cos kl) & (\sinh kl - \sin kl) \\ (\sinh kl + \sin kl) & (\cosh kl - \cos kl) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \quad (3)$$
- Setting the determinant = 0 yields
 
$$\cos kl = \frac{1}{\cosh kl}$$
- Which has roots at
 
$$k_1 l = 4.730 \quad k_2 l = 7.853 \quad k_3 l = 10.996$$
- Substituting (2) into (1) finally yields:
 
$$k^4 = \frac{\rho A}{EI} \omega^2 \rightarrow f_n = \frac{(k_n l)^2}{2\pi l^2} \sqrt{\frac{EI}{\rho A}} \quad \left[ \text{Free-Free Beam Frequency Equation} \right]$$

*These values of  $k_n l$  correspond to the different modes of vibration!*


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**Higher Order Free-Free Beam Modes**

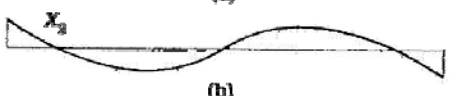
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Mode	$n$	Nodal Points	$k_n l$	$f_n/f_1$
Fundamental ( $f_1$ )	1	2	4.730	1.000
1st Harmonic	2	3	7.853	2.757
2nd Harmonic	3	4	10.996	5.404
3rd Harmonic	4	5	14.137	8.932
4th Harmonic	5	6	17.279	13.344


← More than 10x increase



Fundamental Mode ( $n=1$ )




1st Harmonic ( $n=2$ )



2nd Harmonic ( $n=3$ )

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## Mode Shape Expression


- The mode shape expression can be obtained by using the fact that  $A=C$  and  $B=D$  into (2), yielding

$$u_x = \mathcal{A} \left[ \left( \frac{\mathcal{A}}{\mathcal{B}} \right) (\cosh kx + \cos kx) + (\sinh kx + \sin kx) \right]$$

- Get the amplitude ratio by expanding (3) [the matrix] and solving, which yields

$$\frac{\mathcal{A}}{\mathcal{B}} = \frac{\sin kl - \sinh kl}{\cosh kl - \cos kl}$$

- Then just substitute the roots for each mode to get the expression for mode shape



Fundamental Mode (n=1)  
 [Substitute  $k_1 l = 4.730$ ]

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