**Lecture 14: Beam Bending**

- Announcements:
- HW#4 online and due Tuesday, Oct 16
- Lecture Module 8 online
- Midterm is nearing: Thursday, Oct. 25
  - I will soon pass out materials associated with the midterm, including and information sheet and old exams
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- Reading: Senturia, Chpt. 8
- Lecture Topics:
  - Stress, strain, etc., for isotropic materials
  - Thin films: thermal stress, residual stress, and stress gradients
  - Internal dissipation
  - MEMS material properties and performance metrics
- -------------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- -------------------------------------
- Last Time:
- Went through Module 7 on Mechanics of Materials
- Now finish this
- Then, start a new topic: Bending of Beams

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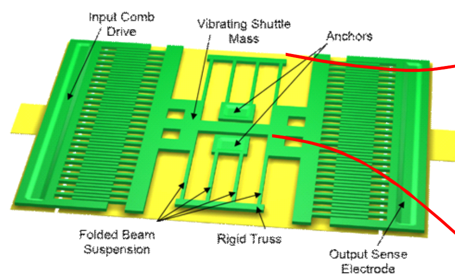
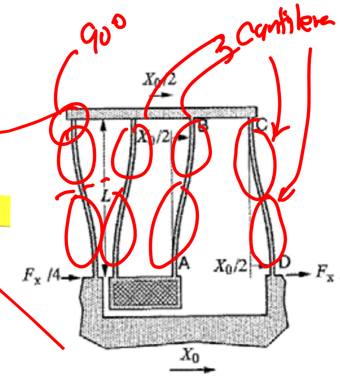
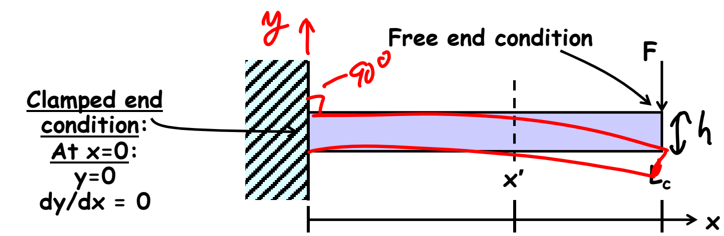
- Springs and suspensions very common in MEMS
- Coils are popular in the macro-world; but not easy to make in the micro-world
- Beams: simpler to fabricate and analyze; become "stronger" on the micro-scale → use beams for MEMS



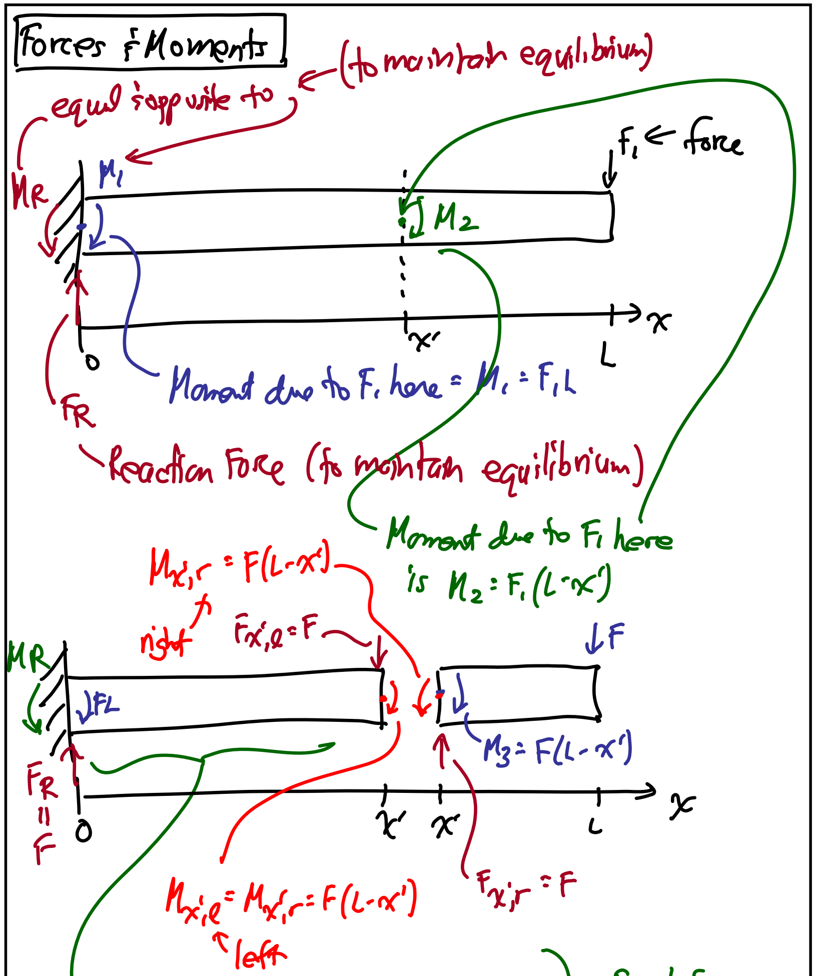




Comb-Driven Folded Beam Actuator

## Problem: Bending a Cantilever Beam



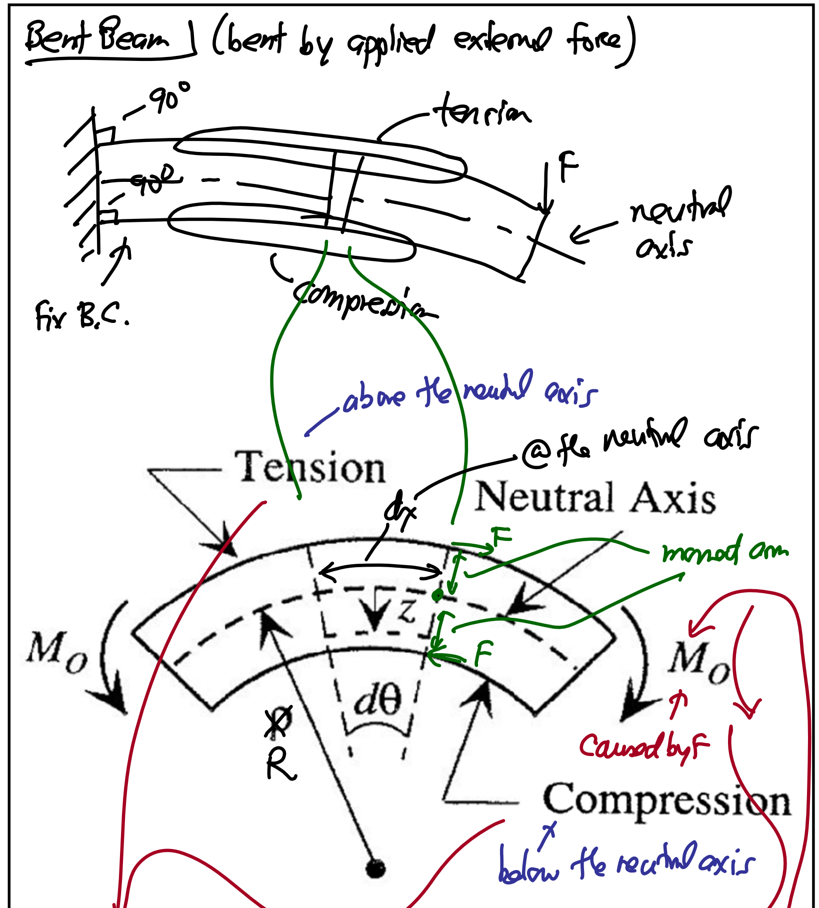

- Objective: Find relation between tip deflection $y(x=L_c)$ and applied load F
- Assumptions:
  1. Tip deflection is small compared with beam length
  2. Plane sections (normal to beam's axis) remain plane and normal during bending, i.e., "pure bending"
  3. Shear stresses are negligible




For left side of beam:

Total Force: $F_R - F_{x',l} = F - F = 0$ ✓

Total Moment: $-M_R + M_{x',l} + x' F_{x',l}$
$= -FL + F(L-x') + x'F = 0$ ✓




Combine to generate forces → moments that fight against the moment due to F

This is how stiffness is generated!

$k_c \triangleq$ stiffness of a cantilever (we want to find this)

## Beam Segment in Pure Bending

⇒ consider the segment bounded by the dashed lines defined by $d\theta$

At $z=0$: neutral axis → segment length $= dx = R\,d\theta$ (1)

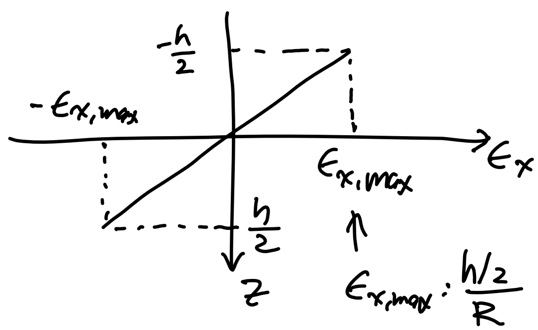
At any $z$: segment length $= dL = dx - z\,d\theta = dx - \frac{z}{R}dx$

Thus, the axial strain @ $z$:

$$\epsilon_x = \frac{dL - dx}{dx} = -\frac{z}{R}$$

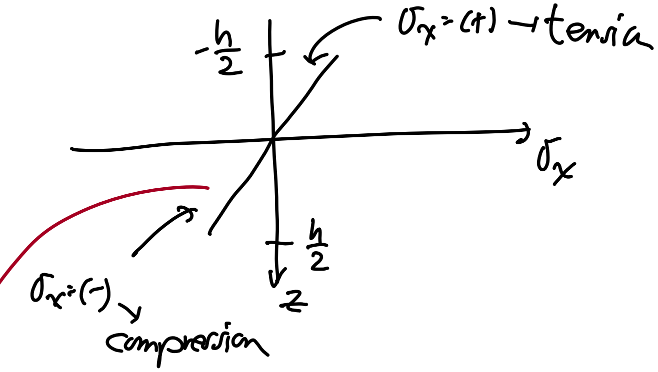
$$\boxed{\epsilon_x = -\frac{z}{R}}$$

Thus, the strain varies linearly along beam thickness:



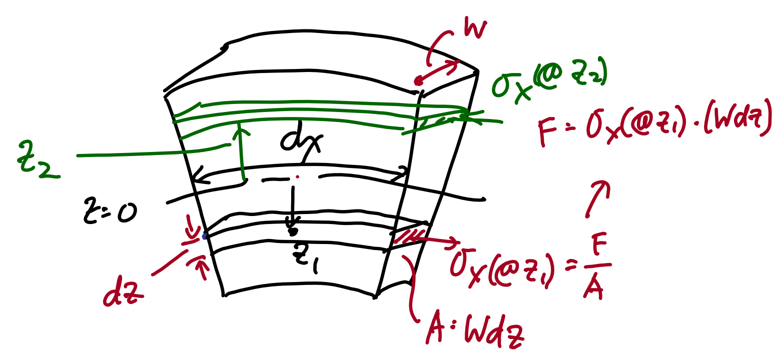

Of course, there is a corresponding axial stress:

$$\sigma_x = \epsilon_x E = \boxed{-\frac{zE}{R} = \sigma_x}$$




This gradient of stress generates a bending moment! → in response to the original applied moment from F

## Stress → Force:



⇒ integrate the stress (or moment) through the thickness of the beam:

$$M = \int_{-\frac{h}{2}}^{\frac{h}{2}} [\underbrace{(W dz)\sigma_x}_{\text{force}}] \cdot z$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{EWz^2}{R} dz \Rightarrow \boxed{M = -\left(\frac{1}{12}Wh^3\right)\frac{E}{R}}$$

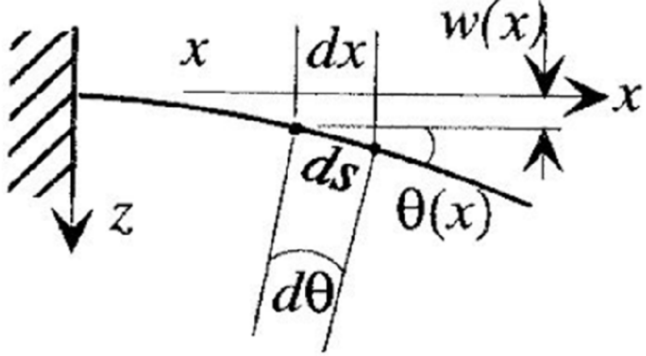
$$\left[\sigma_x = -\frac{zE}{R}\right]$$

$\frac{1}{12}Wh^3 = I \triangleq$ Moment of Inertia

$$\boxed{\frac{1}{R} = -\frac{M}{EI}}$$

Note: (+) radius of curvature ↓ (-) internal bending moment

Differential Equation for Beam Bending




Write out some geometric relationships:
⇒ then use small angle approx

$$\cos\theta = \frac{dx}{ds} \rightarrow ds = \frac{dx}{\cos\theta} \longrightarrow ds \cong dx$$

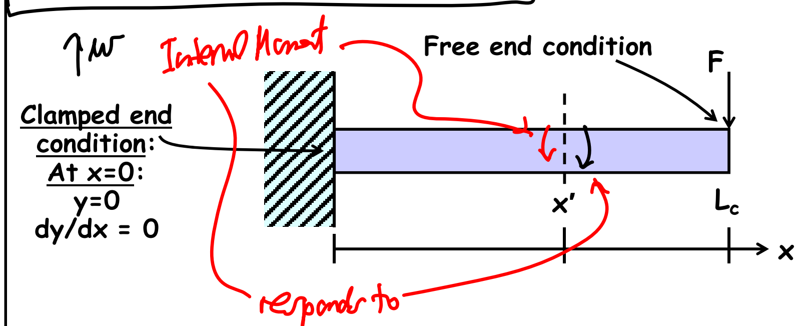
$$\tan\theta = \frac{dw}{dx} = \text{slope of the beam @ this pt.} \longrightarrow \theta \cong \frac{dw}{dx} \quad (1)$$

$$ds = R\,d\theta \rightarrow \frac{1}{R} = \frac{d\theta}{ds} \longrightarrow \frac{1}{R} \cong \frac{d\theta}{dx} \quad (2)$$

Inserting (1) into (2):

$$\frac{1}{R} = \boxed{\frac{d^2w}{dx^2} = -\frac{M}{EI}}$$ ← Diff. Eqn. for Small Angle Beam Bending

Cantilever Beam w/ Concentrated Load




Internal Moment @ position x': $M = -F(L-x)$

Thus: $\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$