Lecture 15: Stress Gradients

- Announcements:
- Lecture Module 9 online
- Midterm is nearing: Thursday, Oct. 25
  - I will soon pass out materials associated with the midterm (info sheet and old exams)
- Delete prob. 4 from HW#4; it will go to HW#5
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- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- -----------------------------------
- Last Time:

$\cos\theta = \frac{dx}{ds} \rightarrow ds = \frac{dx}{\cos\theta} \rightarrow ds \cong dx$

$\tan\theta = \frac{dw}{dx}$ = slope of the beam @ this pt. $\rightarrow \theta \cong \frac{dw}{dx}$ (1)

$ds = R\,d\theta \rightarrow \frac{1}{R} = \frac{d\theta}{ds} \rightarrow \frac{1}{R} \cong \frac{d\theta}{dx}$ (2)

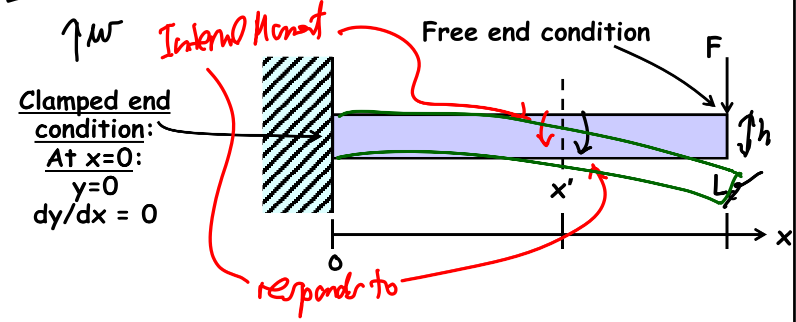
Inserting (1) into (2):

internal bending moment

$$\frac{1}{R} = \boxed{\frac{d^2w}{dx^2} = -\frac{M}{EI}}$$

← Diff. Eqn. for Small Angle Beam Bending

Cantilever Beam w/ Concentrated Load



Internal Moment @ position x': $M = -F(L-x)$

Thus: $\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$

w/ Clamped End B.C.: $w(x=0)=0$, $\frac{dw}{dx}(x=0)=0$

Free End B.C.: none

Solve to get $w = f(x)$

⇒ use Laplace, or a trial solution:

$w = A + Bx + Cx^2 + Dx^3$, then apply B.C.'s.

$$\boxed{w = \frac{FL}{2EI}x^2\left(1 - \frac{x}{3L}\right)}$$

deflection @ x due to a point load F applied @ x=L

Maximum Deflection → occurs @ $x=L$:

$$w_{max} = \left(\frac{L^3}{3EI}\right)F \rightarrow F = \left(\frac{3EI}{L^3}\right) w(x=L)$$

$$= k_c w(x=L)$$

↑ stiffness @ location $x=L$

$$k_c = \frac{3EI}{L^3} \triangleq \text{stiffness}$$

$$\left[I = \frac{1}{12}Wh^3\right] \rightarrow \boxed{k_c = \frac{1}{4}EW\frac{h^3}{L^3}}$$

↑ width

Ex. $L=100\mu m, W=2\mu m, h=2\mu m$

polysilicon → $E = 150\,GPa$

$$k_c = \frac{1}{4}(150G)(2\mu)\left(\frac{2\mu}{100\mu}\right)^3 = 0.6\,N/m$$

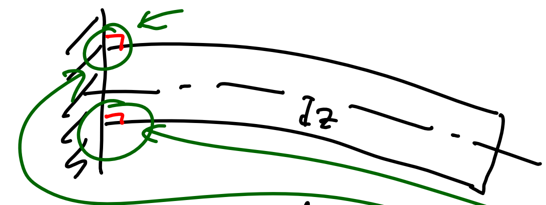
Maximum Stress in a Bent Condition

From before, the radius of curvature is given by

$$\frac{1}{R} = \frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

⇒ $\frac{1}{R}$ is maximized (i.e., R is minimized) when $x=0$:

$$[x=0] \rightarrow \frac{1}{R} = \frac{d^2w}{dx^2} = \frac{FL}{EI}$$



Strain is maximized:

① At top surface → tensile

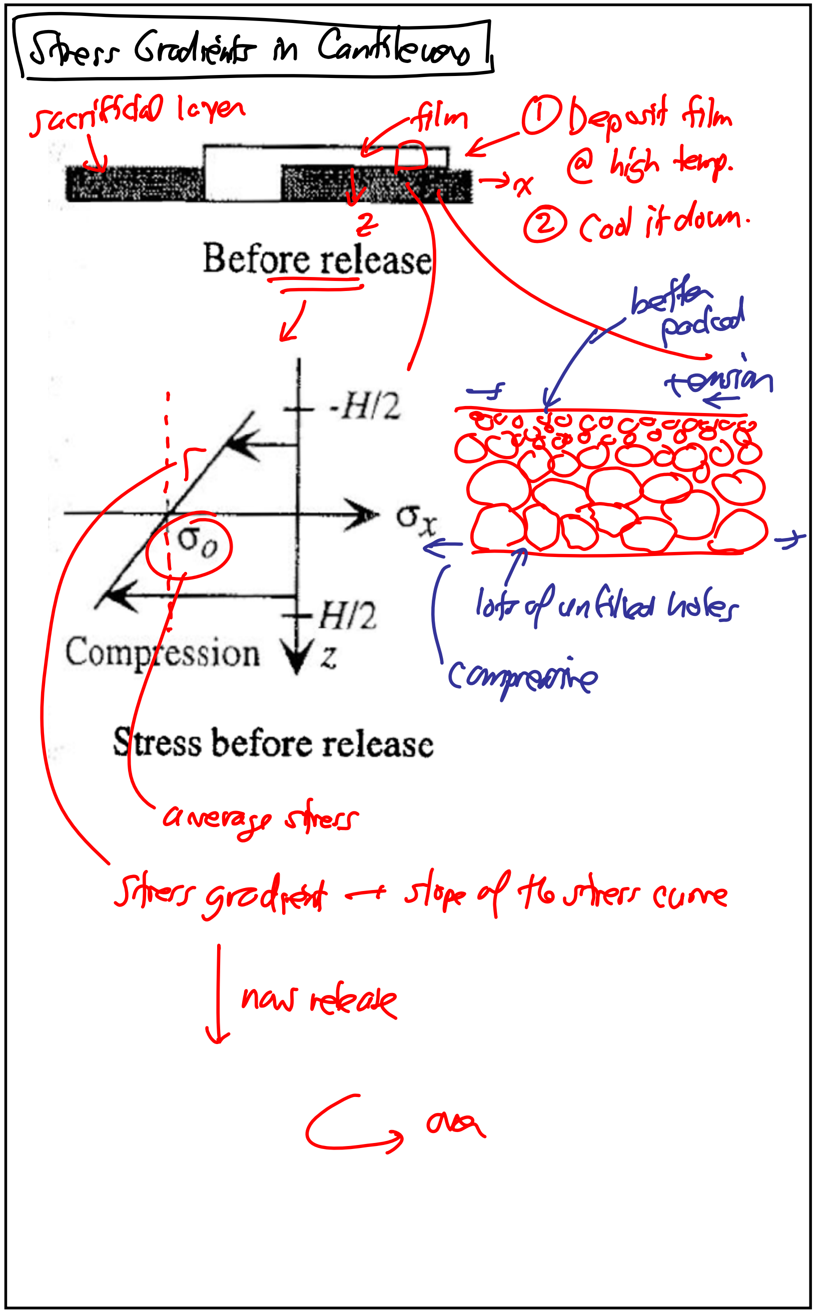
② At bottom surface → compressive

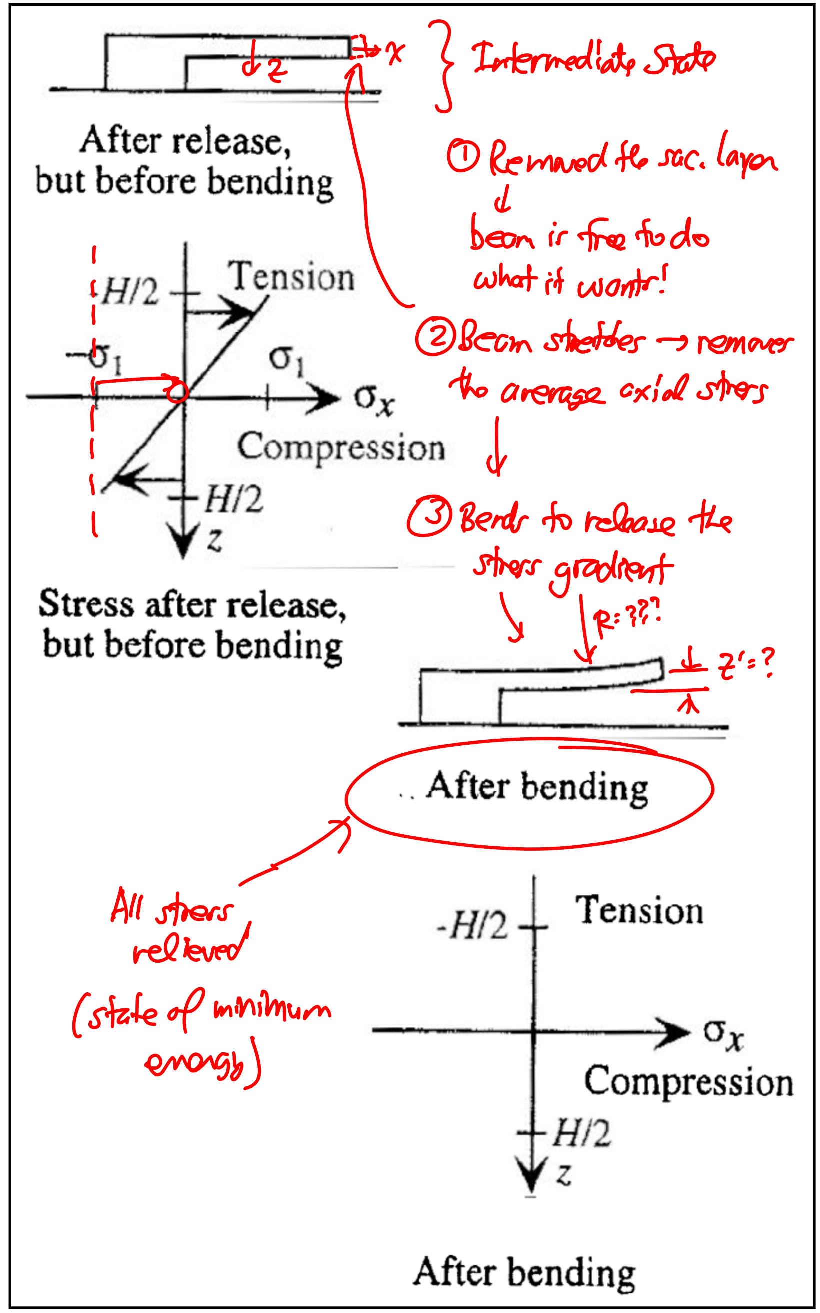
$$\epsilon_{max} = \frac{z}{R} = \frac{h}{2}\frac{1}{R} = \frac{h}{2}\frac{FL}{EI} = \epsilon_{max}$$

$$\left[I = \frac{1}{12}Wh^3\right] \Rightarrow \epsilon_{max} = \frac{\cancel{h}}{2}\frac{FL}{E}\left(\frac{12}{Wh^{\cancel{3}\,2}}\right) = \frac{6L}{EWh^2}F$$

$$\boxed{\sigma_{max} = \epsilon_{max}E = \frac{6L}{Wh^2}F}$$

(maximum stress in a bent condition subjected to a force F at its tip)


Stress Gradients in Cantilevers
sacrificial layer
film
① Deposit film @ high temp.
② Cool it down.
x
z
Before release
-H/2
σx
σ0
H/2
z
Compression
Stress before release
better packed
tension
lots of unfilled holes
compressive
average stress
Stress gradient → slope of the stress curve
now release


Intermediate State
x
z
After release, but before bending
① Removed the sac. layer
beam is free to do what it wants!
② Beam stretches → removes the average axial stress
③ Bends to release the stress gradient
-H/2
Tension
-σ1
σ1
σx
Compression
H/2
z
Stress after release, but before bending
R=??
z'=?
After bending
All stress relieved (state of minimum energy)
-H/2
Tension
σx
Compression
H/2
z
After bending

Bending Due to Stress Gradients

Find the radius of curvature:

Prior to release, axial stress: $\sigma = \sigma_0 - \frac{\sigma_1}{(H/2)} z$

The internal moment:

$$M_x = \int_{-\frac{H}{2}}^{\frac{H}{2}} [(w\,dz)\cdot\sigma]\cdot z = \int_{-\frac{H}{2}}^{\frac{H}{2}} w\left(z\sigma_0 - \frac{\sigma_1 z^2}{(H/2)}\right)dz$$

$$= w\left(\frac{1}{2}\sigma_0 z^2 - \frac{2\sigma_1 z^3}{3H}\right)\Bigg|_{-H/2}^{H/2}$$

$$= w\left(\cancel{\frac{1}{2}\sigma_0\frac{H^2}{4}} - \frac{2}{3}\sigma_1\frac{H^2}{8} - \cancel{\frac{1}{2}\sigma_0\frac{H^2}{4}} - \frac{2}{3}\sigma_1\frac{H^2}{8}\right)$$

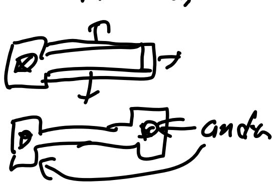
$$\boxed{M_x = -\frac{1}{6}\sigma_1 w H^2}$$

Thus, the radius of curvature:

$$\frac{1}{R} = -\frac{M_x}{E'I} \rightarrow R = \frac{E'I}{M_x} = \frac{1}{2}\frac{E'H}{\sigma_1}$$

Biaxial Modulus

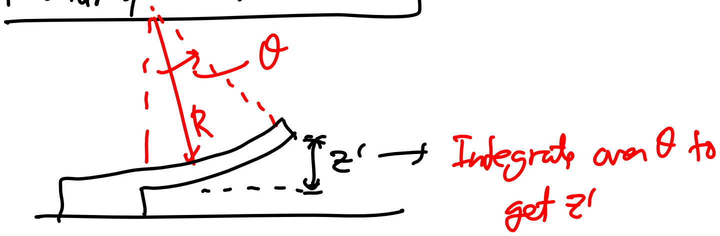
$$\left[I = \frac{1}{12}wH^3\right]$$

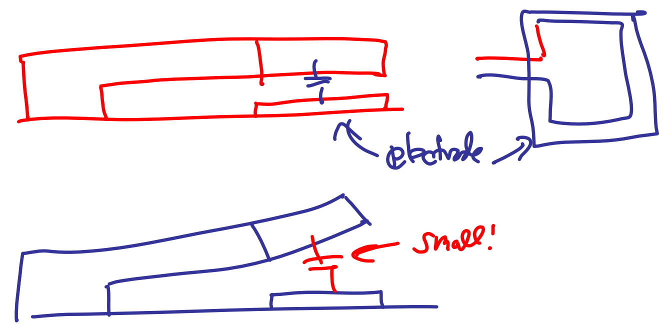


$$\boxed{R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1}}$$

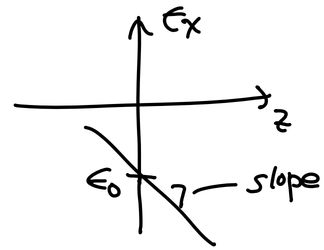
Radius of Curvature for a Cantilever w/ a Stress Gradient

Radius of Curvature → z'





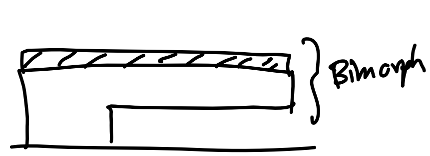
Definition. Strain Gradient

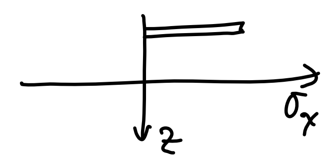


slope ≙ Strain Gradient = Γ

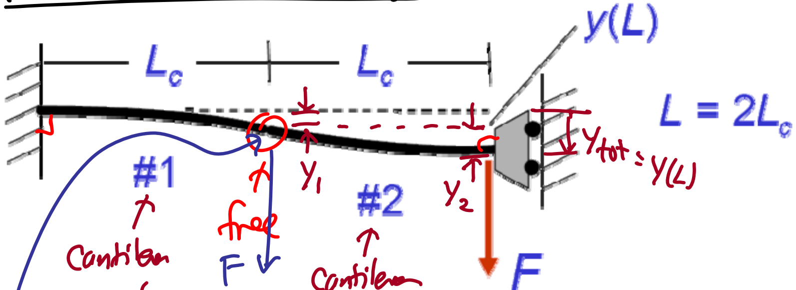
$$\Gamma = \frac{\epsilon_1}{(H/2)}$$

$$R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1} = \frac{H}{2}\frac{E'}{\sigma_1} = \frac{(H/2)}{\epsilon_1} = \frac{1}{\Gamma} \rightarrow \boxed{\Gamma = \frac{1}{R}}$$





$\boxed{\text{Series Combination of Springs}}$



identical due to symmetry

Series: $y_{tot} = y_1 + y_2$ → in series if $y_{tot} > y_1 = y(L_c)$ and $y_{tot} > y_2$

What is the force here? → F

$$y(L) = \frac{F}{k_{tot}} = 2y(L_c) = 2\left(\frac{F}{k_c}\right) = F\left(\frac{1}{k_c}+\frac{1}{k_c}\right)$$

$$k_c = \frac{1}{4}EW\left(\frac{h}{L_c}\right)^3$$

$$\frac{1}{k_{tot}} = \frac{1}{k_c}+\frac{1}{k_c} \rightarrow \boxed{k_{tot} = k_c \| k_c}$$

Definition for "||" : $\left\{ A\|B = \frac{1}{\frac{1}{A}+\frac{1}{B}} = \frac{AB}{A+B} \right.$