Lecture 17: Beam Combos II

- Announcements:
- Midterm in one week: Thursday, Oct. 25
  - I passed out an information sheet and old exams for midterm preparation
- HW#5 online .. due Tuesday next week
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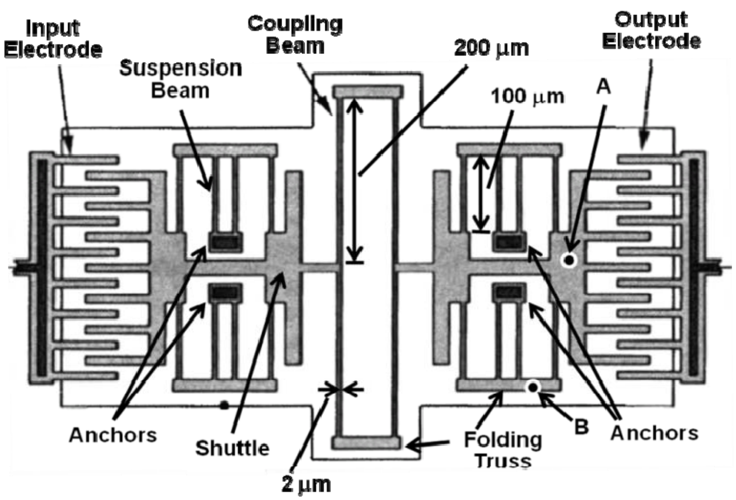
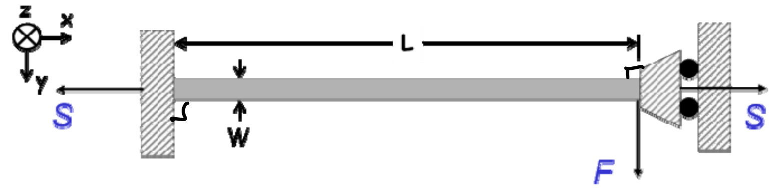
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- ------------------------------------
- Last Time:
- Finished beam combo mechanical spring circuits
- Now, start into the effects of residual stress



Tensioned Spring (Non-Ideality)

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L



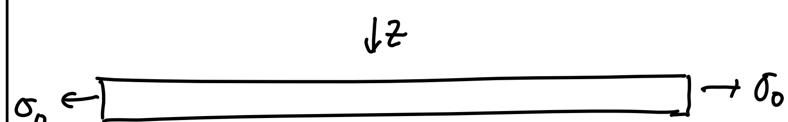
Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4 y}{dx^4} - S\frac{d^2 y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

Axial Load

Heuristic Derivation for the Euler Beam Eqn.

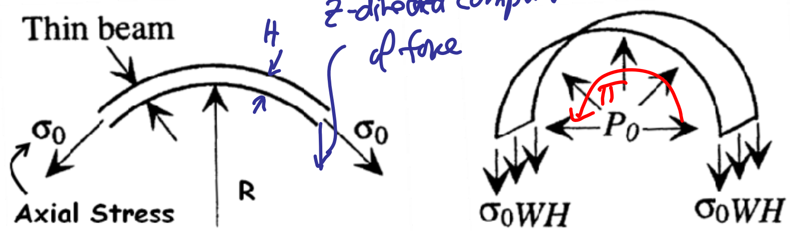
Consider first a straight beam under axial stress:



⇒ no effect on z-directed forces ∴ no effect on the z-directed stiffness when the beam is straight

...but, when the beam is bent:

z-directed component of force → now k_z is affected



To model this:

① Upward pressure P_o to counteract the downward force from σ_o (the tensile stress) to keep everything in equilibrium.

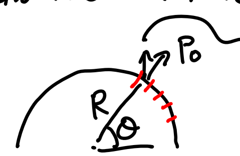
② Then equate terms in equilibrium.

For ease of analysis:

→ Assume beam bent to an angle π

↳ Downward vertical force: $\underline{\underline{2\sigma_o WH}}$

Upward force due to P_o:



$P_y(\theta) = P_o \sin\theta$

$$F_u = \int_0^{\pi} (P_o \sin\theta) W (R\,d\theta)$$

$$= -P_o W R \cos\theta \Big|_0^{\pi}$$

$$= 2RWP_o$$

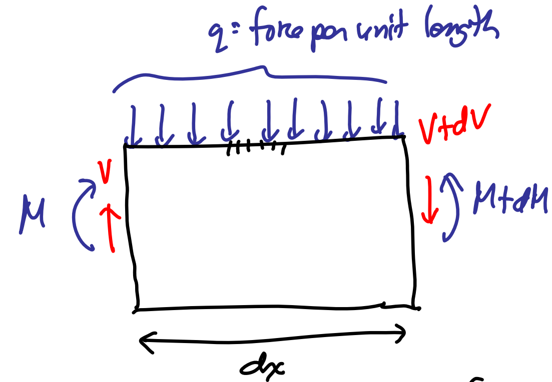
[Equilibrium] → $2RWP_o = 2\sigma_o WH$ → $P_o = \dfrac{\sigma_o H}{R}$

[$q_o = \dfrac{\text{beam load}}{\text{unit length}} = P_o W$, $\dfrac{1}{R} = \dfrac{d^2 w}{dx^2}$] beam displacement

$$q_o = \sigma_o WH \frac{d^2 w}{dx^2}$$ ← generalize to to case of smaller displacements & angles

[use our differential beam bending equation] $\dfrac{d^2 w}{dx^2} = -\dfrac{M}{EI}$ → ??? ——— *

Relationships between forces on a fully loaded differential beam element:



[Total Static Equilibrium] → total force = 0

F_T: total force $= q\,dx + (\cancel{V} + dV) - \cancel{V} = 0$

$$\therefore \boxed{\frac{dV}{dx} = -q} \qquad (1)$$

→ also, total moment w/r to left side = 0

$$M_T = (\cancel{M} + dM) - \cancel{M} - (V + \cancel{dV})dx - \tfrac{1}{2} q \cancel{dx^2} = 0$$

[Neglect products of differentials] $\left[\int_0^{dx} (q\,du)\,u = \tfrac{1}{2} q\,dx^2 \right]$

$$dM - V dx = 0 \longrightarrow \boxed{\frac{dM}{dx} = V} \qquad (2)$$

Combining (1) & (2):

$$\left[\frac{d^2M}{dx^2} = \frac{dV}{dx} = -q\right]$$

load / unit length

$$* \longrightarrow \frac{d^2w}{dx^2} = -\frac{M}{EI} \longrightarrow \frac{d^4w}{dx^4} = \frac{q}{EI}$$

$$EI\frac{d^4w}{dx^4} = q + q_0$$

external load

equiv. load f/ axial stress

$$\left[q_0 = \sigma_0 WH \frac{d^2w}{dx^2}\right]$$

$$\boxed{EI\frac{d^4w}{dx^2} - \underbrace{(\sigma_0 WH)}_{\text{tension in the beam} = S}\frac{d^2w}{dx} = q}$$

force

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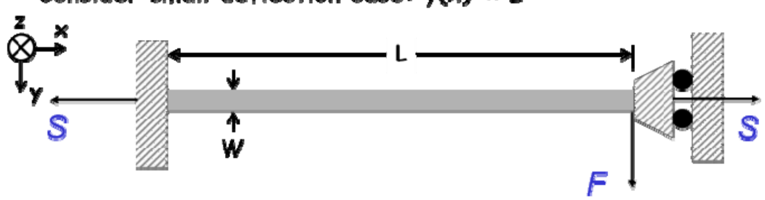
- Consider small deflection case: y(x) « L



Governing differential equation: (Euler Beam Equation)

$$EI_z\frac{d^4y}{dx^4} - \underset{\text{Axial Load}}{S}\frac{d^2y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

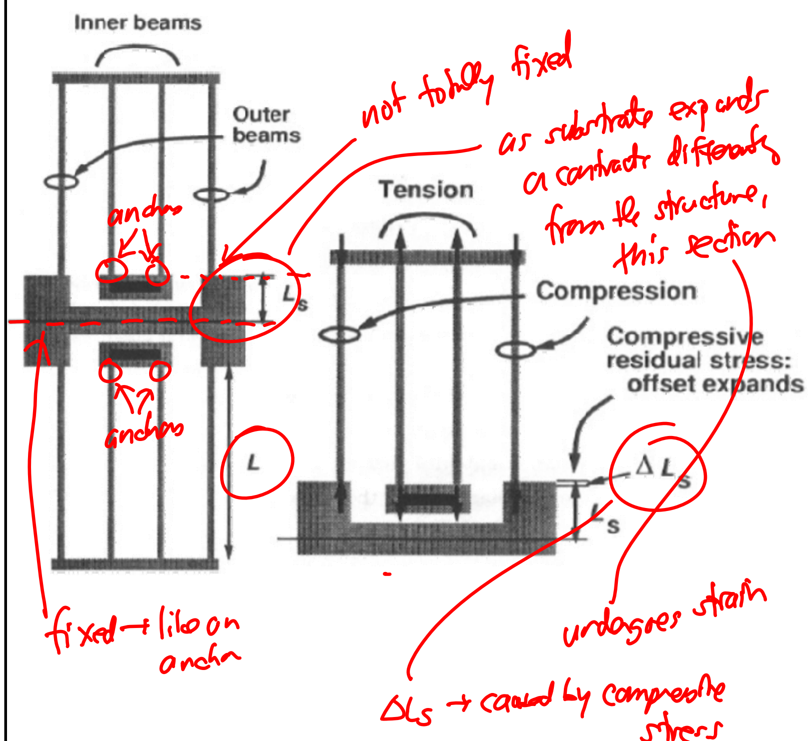
- Can solve the ODE using standard methods
  - Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)
  - For solution to the clamped-guided case: see S. Timoshenko, Strength of Materials II: Advanced Theory and Problems, McGraw-Hill, New York, 3rd Ed., 1955
- Result from Timoshenko:

S > 0 (tension) $\quad k^{-1} = \frac{pL - 2\tanh(pL/2)}{p|S|} = \frac{y(x=L)}{F}$

S < 0 (compression)

$$k^{-1} = \frac{-pL + 2\tan(pL/2)}{p|S|} = \frac{y(x-L)}{F}$$

where $p = \sqrt{\frac{|S|}{EI_z}}$



Determine the Spring Constants (under ΔL_s)

① If polySi strain is ϵ_r, then the shoulder expands by $\Delta L_s = \epsilon_r L_s$.

② This then applies a load to the beams → $\Delta L = \Delta L_s$

③ Beam Stress:

$$\epsilon_b = \frac{\Delta L}{2L} = \frac{\Delta L_s}{2L} = \pm \epsilon_r \frac{L_s}{2L}$$

Stress Force: $S = \pm E \epsilon_r \left(\frac{L_s}{2L}\right) Wh$ (axial tension)

④ Spring Constants: $k_{com} \parallel k_{ten}$

in series

$$k = 4(k_{com}^{-1} + k_{ten}^{-1})^{-1}$$

$$k = 4\left[\frac{-pL + 2\tan(pL/2)}{pS|} + \frac{pL - 2\tanh(pL/2)}{p|S|}\right]^{-1}$$

4 in parallel