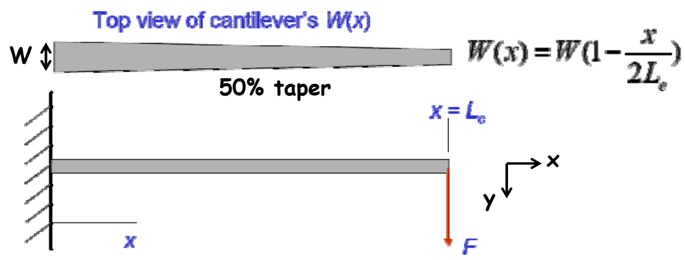
More General Geometries

- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- Example: tapered cantilever beam
- Objective: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)



Solution: Use Principle of Virtual Work

- In an energy-conserving system (i.e., elastic materials), the energy stored in a body due to the quasi-static (i.e., slow) action of surface and body forces is equal to the work done by these forces …
- Implication: if we can formulate stored energy as a function of the deformation of a mechanical object, then we can determine how an object responds to a force by determining the shape the object must take in order to minimize the difference U between the stored energy and the work done by the forces:

U = Stored Energy - Work Done

- Key idea: we don't have to reach U = 0 to produce a very useful, approximate *analytical* result for load-deflection

Shear Strain Energy

$$\boxed{W_{shear} = \frac{3(EI_z)^2}{4GWh}\int_0^L \left(\frac{d^3y}{dx^3}\right)^2 dx}$$

Shear Modulus

- See W.C. Albert, "Vibrating Quartz Crystal Beam Accelerometer," Proc. ISA Int. Instrumentation Symp., May 1982, pp. 33-44

Applying the Principle of Virtual Work

- Basic Procedure:
  - Guess the form of the beam deflection under the applied loads
  - Vary the parameters in the beam deflection function in order to minimize:

Assumes point load

$$U = \overbrace{\sum_j W_j}^{\text{Sum strain energies}} - \sum_i F_i u_i$$

Displacement at point load

  - Find minima by simply setting derivatives to zero
- See Senturia, pg. 244, for a general expression with distributed surface loads and body forces

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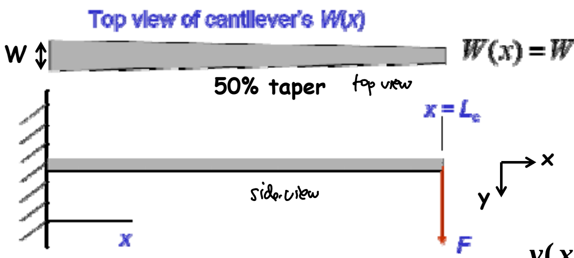


$$W(x) = W(1 - \frac{x}{2L_c})$$

Adjustable parameters: minimize U

- Start by guessing the solution → $y(x) = c_2 x^2 + c_3 x^3$
  - It should satisfy the boundary conditions
  - The strain energy integrals shouldn't be too tedious
    - This might not matter much these days, though, since one could just use matlab or mathematica

Strain Energy And Work By F

$$U = W_{bend} - F \cdot y(L_c)$$

$$W_{bend} = \frac{1}{2} E \int_0^{L_c} I_z(x) \left(\frac{d^2 y}{dx^2} \right)^2 dx \quad \text{(Bending Energy)}$$

$$I_z(x) = \frac{W(x) h^3}{12} \qquad \frac{d^2 y}{dx^2} = 2c_2 + 6c_3 x \quad \text{(Using our guess)}$$

$$W(x) = W(1 - \frac{x}{2L_c})$$

$$= \frac{1}{24} E W h^3 \int_0^{L_c} (1 - \frac{x}{2L_c})(2c_2 + 6c_3 x)^2 dx - \underbrace{F(c_2 L_c^2 + c_3 L_c^3)}_{\text{Tip Deflection}}$$

Find c_2 and c_3 That Minimize U

- Minimize U → basically, find the c_2 and c_3 that brings U closest to zero (which is what it would be if we had guessed correctly)
- The c_2 and c_3 that minimize U are the ones for which the partial derivatives of U with respective to them are zero:

$$\frac{\partial U}{\partial c_2} = 0 \qquad \frac{\partial U}{\partial c_3} = 0$$

- Proceed:
  - First, evaluate the integral to get an expression for U:

$$U = EWh^3 \left\{ \frac{5c_3^2}{16} L_c^3 + \frac{c_2 c_3}{3} L_c^2 + \frac{c_2^2}{8} L_c \right\} - F\left(c_2 L_c^2 + c_3 L_c^3\right)$$

Minimize U (cont)

- Evaluate the derivatives and set to zero:

$$\frac{\partial U}{\partial c_2} = 0 = \left(\frac{EWh^3}{3} c_3 - F \right) L_c^2 + \left(\frac{EWh^3}{4} c_2 \right) L_c$$

$$\frac{\partial U}{\partial c_3} = 0 = \left(\frac{5}{8} EWh^3 c_3 - F \right) L_c^3 + \left(\frac{EWh^3}{3} c_2 \right) L_c^2$$

- Solve the simultaneous equations to get c_2 and c_3:

$$\boxed{c_2 = \left(\frac{84}{13} \right) \frac{F L_c}{EWh^3} \qquad c_3 = -\left(\frac{24}{13} \right) \frac{F}{EWh^3}}$$

The Virtual Work-Derived Solution

- And the solution:

$$y(x) = \left(\frac{24F}{13EWh^3}\right)\left(\left(\frac{7}{2}\right)L_c - x\right)x^2$$

- Solve for tip deflection and obtain the spring constant:

$$y(L_c) = \left(\frac{24F}{13EWh^3}\right)\left(\frac{5}{2}\right)L_c^{\,3} \qquad k_c = F / y(L_c) = \left(\frac{13EWh^3}{60L_c^{\,3}}\right)$$

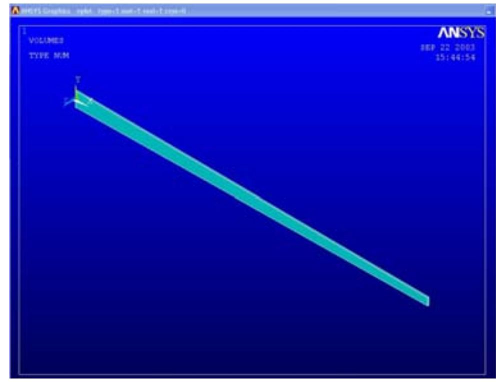
- Compare with previous solution for constant-width cantilever beam (using Euler theory):

$$y(L_c) = \left(\frac{4F}{EWh^3}\right)L_c^{\,3} \longrightarrow$$ 13% smaller than tapered-width case

Comparison With Finite Element Simulation

- Below: ANSYS finite element model with

L = 500 μm W_{base} = 20 μm E = 170 GPa
h = 2 μm W_{tip} = 10 μm



- Result: (from static analysis)
  - k = 0.471 μN/m
- This matches the result from energy minimization to 3 significant figures

Need a Better Approximation?

- Add more terms to the polynomial
- Add other strain energy terms:
  - Shear: more significant as the beam gets shorter
  - Axial: more significant as deflections become larger
- Both of the above remedies make the math more complex, so encourage the use of math software, such as Mathematica, Matlab, or Maple
- Finite element analysis is really just energy minimization
- If this is the case, then why ever use energy minimization analytically (i.e., by hand)?
  - Analytical expressions, even approximate ones, give insight into parameter dependencies that FEA cannot
  - Can compare the importance of different terms
  - Should use in tandem with FEA for design