Lecture 18: Energy Methods

- Announcements:
- Midterm is nearing: Thursday, Oct. 25
  - I passed out materials associated with the midterm, including and information sheet and old exams, last Thursday
- HW#5 due today
  - Solutions will be posted tonight (or emailed)
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- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
  - Estimating Resonance Frequency
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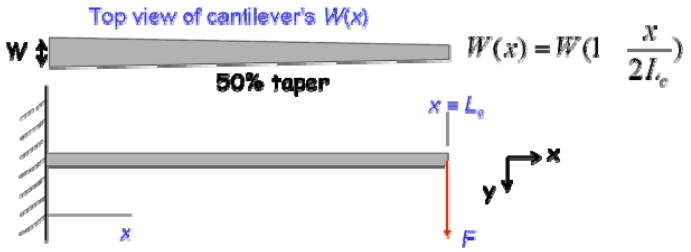
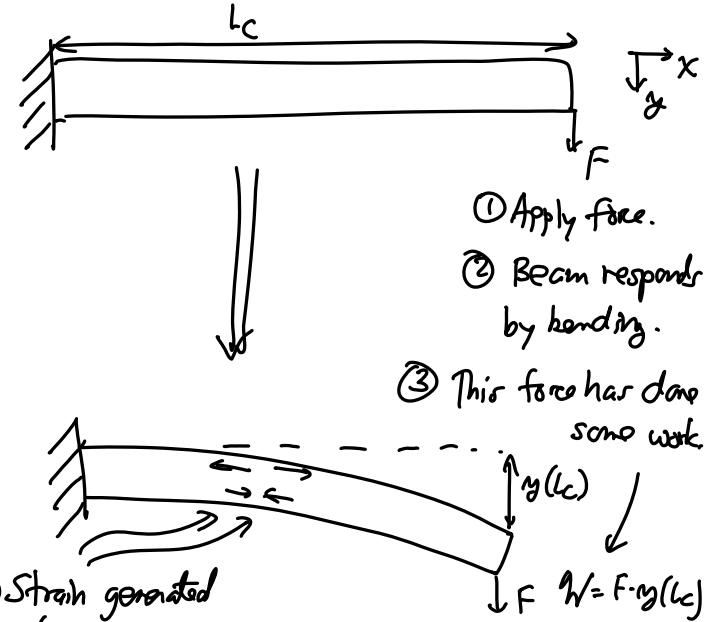
- Last Time: Finished beam combos

## More General Geometries

- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- Example: tapered cantilever beam
- Objective: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)




Same problem as before: Cantilever beam w/ force applied (@ its tip)




① Apply force.

② Beam responds by bending.

③ This force has done some work.

④ Strain generated
↳ So the beam has received an influx of stored energy
↳ magnitude of " " determined by shape

$W = F \cdot y(L_c)$

⑤ Then:

U = Stored Energy − Work Done → 0

↑ When we choose the right shape

↳ From this, get expression for $y = f(x,F)$

## Fundamentals of Energy Density

### General Definition of Work

$$W(q_1) = \int_0^{q_1} e(q)\,dq$$

$q$ = displacement
$e$ = effort

↳ for EE: $W(Q) = \int_0^Q \frac{Q}{C}\,dQ$

### Strain Energy Density

$$w = \int_0^{\epsilon_x} \sigma_x\,d\epsilon_x$$

$\epsilon_x$ ← value of strain @ position $(x, y, z)$

$\sigma_x(\epsilon_x)$ → relates stress to strain @ position $(x, y, z)$

$[\sigma_x = E\epsilon_x]$ ↓

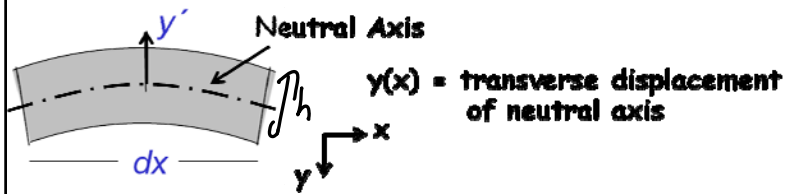
$$w = \int_0^{\epsilon_x} E\epsilon_x\,d\epsilon_x = \frac{1}{2}E\epsilon_x^2$$

### Total Strain Energy: [J]

$$W = \iiint \left[\frac{1}{2}E\left(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2\right) + \frac{1}{2}G\left(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2\right)\right] dV$$

$dV$ ← volume

## Bending Energy Density




First, find the bending energy $dW_{bend}$ in an infinitesimal length $dx$:

$$dW_{bend} = W\,dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\epsilon_x^2(y')\,dy'$$

($W$ ← width (out of the plane of the page))

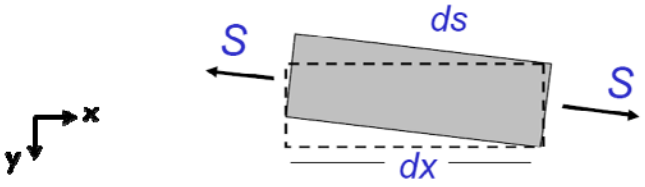
$$\left[\frac{1}{R} = \frac{d^2y}{dx^2},\ \epsilon_x = \frac{y'}{R}\right] \Rightarrow \epsilon_x(y') = y'\frac{d^2y}{dx^2}$$

$$dW_{bend} = W\,dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\left[y'\frac{d^2y}{dx^2}\right]^2 dy'$$

$$= \frac{1}{2}E\underbrace{\left(\frac{Wh^3}{12}\right)}_{I_z}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\therefore \boxed{W_{bend} = \frac{1}{2}EI_z\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx}$$

## Energy Due to Axial Load




⇒ energy related to lengthening:

$$ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

Binomial Theorem → $\simeq dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$

$$\therefore \epsilon_x = \frac{ds - dx}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

$$\left[dW_{axial} = S\epsilon_x dx = \frac{1}{2}S\left(\frac{dy}{dx}\right)^2 dx\right]$$

$$W_{axial} = \frac{1}{2}S\int_0^L \left(\frac{dy}{dx}\right)^2 dx$$

↑ Axial Strain Energy

⇒ Look @ shear strain energy in your module.

- Go through Module 9 pages 10-18.

## Estimating Resonance Frequency


$$\mapsto x(t) = X_0 \cos\omega t$$

**Potential Energy**

$$W(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kX_0^2\cos^2\omega t$$

**Kinetic Energy**

$$K(t) = \frac{1}{2}M\dot{x}(t) = \frac{1}{2}MX_0^2\omega^2\sin^2\omega t$$

$$\left[\dot{x} = \frac{dx}{dt} = \text{velocity}\right]$$

**Remarks:**

① Energy must be conserved.

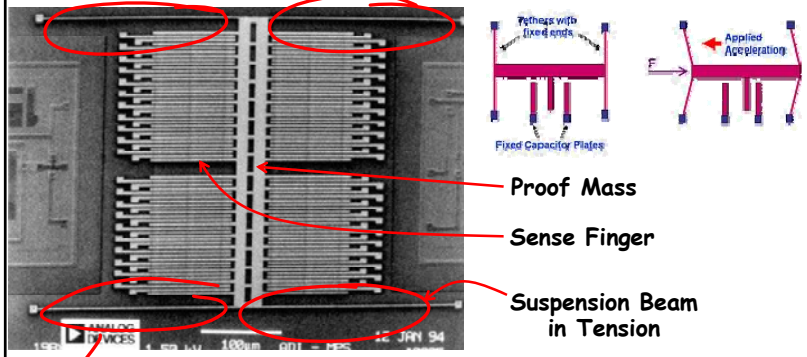
② Total Energy = Potential Energy + Kinetic Energy
(at all times & location on the structure)

$$W_{max} = \frac{1}{2}kX_0^2 = K_{max} = \frac{1}{2}M\omega^2X_0^2$$

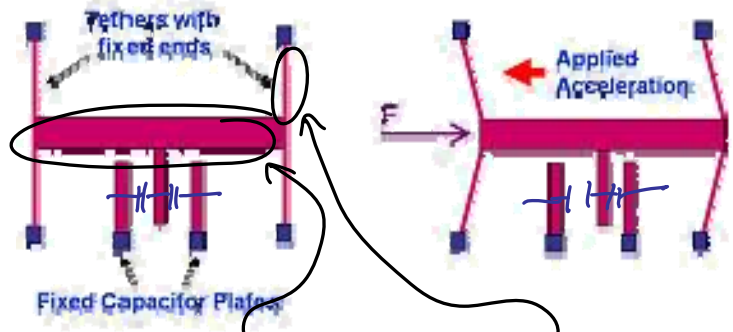
$W_{max}$: maximum potential energy
$X_0$: peak displacement
$K_{max}$: maximum kinetic energy
$\omega$: radian frequency

$\omega_0 = \sqrt{\frac{k}{M}}$ ⇒ good for problems where mass & stiffness can be separated, i.e., are distinct!

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm




In fabrication, purposely introduce tensile stress in the suspension beams!
↑ significant one!

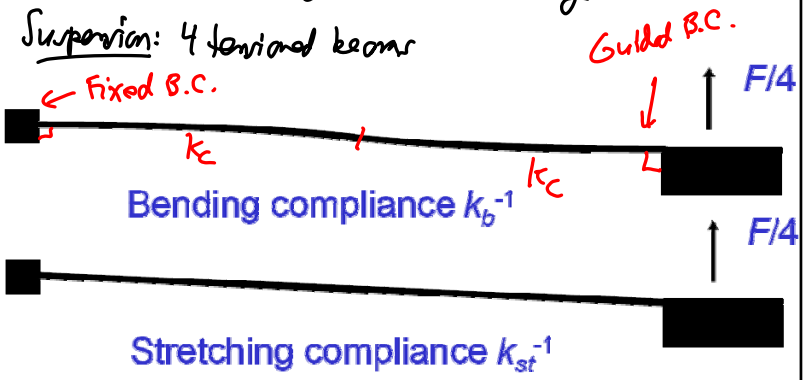



mass of structure >> mass of springs
∴ ignore the mass of the springs

stiffness of springs << stiffness of structure
∴ ignore the stiffness of the structure

For the ADXL-50: 60% of the mass from sense fingers → M = 162 ng

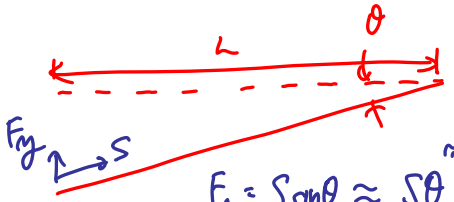
Suspension: 4 tensioned beams



**Bending Contribution**

$$k_b^{-1} = \left(\frac{1}{k_c} + \frac{1}{k_c}\right) = 2\left[\frac{(L/2)^2}{3E(wh^3/12)}\right] = \frac{L^3}{Ewh^3} = 4.2\,\mu m/N$$

**Stretching Contribution**




$$F_y = S\sin\theta \approx S\theta \approx S\left(\frac{y}{L}\right) = \left(\frac{S}{L}\right)y$$

($\sin\theta \approx \tan\theta$; assume small displacements)

($\frac{S}{L}$ = $k_{st}$)

$$k_{st}^{-1} = \frac{L}{S} = \frac{L}{\sigma_r wh} = 1.14\,\mu m/\mu N$$

To get the total spring constant, add bending stiffness to the stretching:

$$k = 4(k_b + k_{st}) = 4(0.24 + 0.88) = 4.5\,\mu N/\mu m$$

Now, get resonance freq.:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4.48\,N/m}{162\times10^{-12}\,kg}} = 26.5\,kHz$$

ADXL-50 Data Sheet: $f_o = 24\,kHz$ ← difference?

→ capacitive transducers

→ electrical stiffness