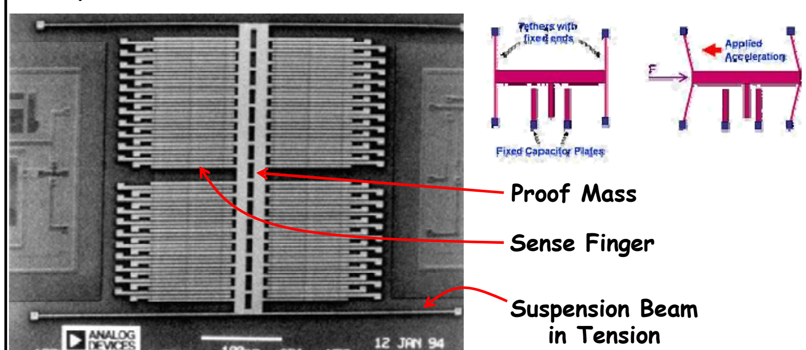
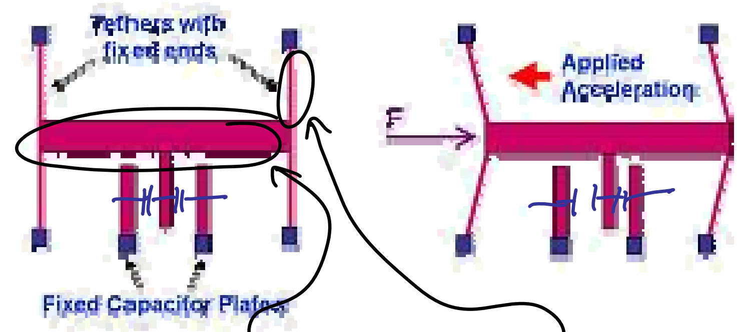
Lecture 19: Resonance Frequency

- Announcements:
- HW#6 will be online soon
- Pass out project today (near end of class)
- Pass back graded midterms today and discussing grading (near end of class)
- ---------------------------------------
- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
- ---------------------------------------
- Last Time:

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm





mass of structure >> mass of springs

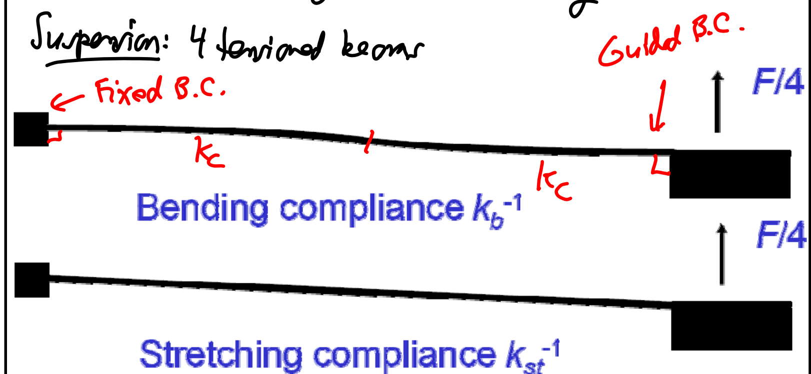
∴ ignore the mass of the springs

stiffness of springs << stiffness of structure

∴ ignore the stiffness of the structure

For the ADXL-50: 60% of the mass from sense fingers → M = 162 ng

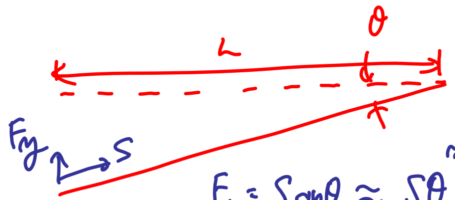
Suspension: 4 tensioned beams



Bending Contribution

$$k_b^{-1} = \left(\frac{1}{k_c} + \frac{1}{k_c}\right) = 2\left[\frac{(L/2)^2}{3E(wh^3/12)}\right] = \frac{L^3}{Ewh^3} = 4.2\,\mu m/N$$

Stretching Contribution



$$F_y = S\sin\theta \approx S\theta \approx S\left(\frac{y}{L}\right) = \underbrace{\left(\frac{S}{L}\right)}_{k_{st}} y$$

$\approx \tan\theta$

[assume small displacements]

$$k_{st}^{-1} = \frac{L}{S} = \frac{L}{\sigma_r wh} = 1.14\,\mu m/\mu N$$

To get the total spring constant, add bending stiffness to the stretching:

$$k = 4(k_b + k_{st}) = 4(0.24 + 0.88) = 4.5\,\mu N/\mu m$$

Now, get resonance freq.:

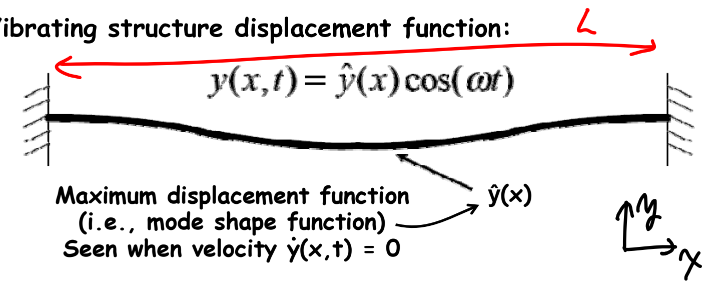
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4.48\,N/m}{162\times10^{-12}\,kg}} = 26.5\,kHz$$

ADXL-50 Data Sheet: $f_o = 24\,kHz$ ← difference?

→ capacitive transducers → electrical stiffness

Find Resonance Frequency When Mass & Stiffness Are Distributed

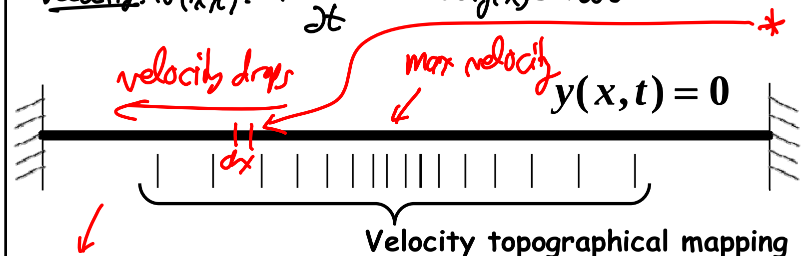
- Vibrating structure displacement function:



- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy W_{max} at the point of maximum displacement (e.g., when $t=0$, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency ← Rayleigh-Ritz

Get Maximum Kinetic Energy

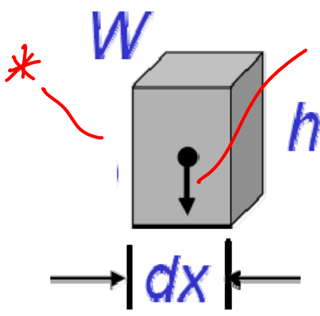
velocity: $v(x,t) = \frac{\partial y(x,t)}{\partial t} = -\omega\hat{y}(x)\sin\omega t$



When $y(x,t)=0$, all the energy in the structure is kinetic ($W=0$)

$$v\left(x, \frac{(2m+1)\pi}{2\omega}\right) = -\omega\hat{y}(x)$$

← $t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \ldots$



velocity:

$v_z = \omega \hat{y}(x)$

$\frac{(2m+1)\pi}{2W}$

$dK = \frac{1}{2} \cdot dm \cdot [v(x,t)]^2$

$dm = \rho (W h dx)$

density

Maximum K.E. -

$$KE_{max} = \int_0^L \frac{1}{2} \rho (W h dx) v^2(x,t) = \int_0^L \frac{1}{2} \rho W h \omega^2 [\hat{y}(x)]^2 dx$$

To get frequency:

$KE_{max} = PE_{max} = W_{max}$

Rayleigh-Ritz formula

$$\omega = \sqrt{\frac{W_{max}}{\int_0^L \frac{1}{2} \rho W h [\hat{y}(x)]^2 dx}}$$

[radians/s]

ω = radian resonance freq.

W_{max} = maximum potential energy

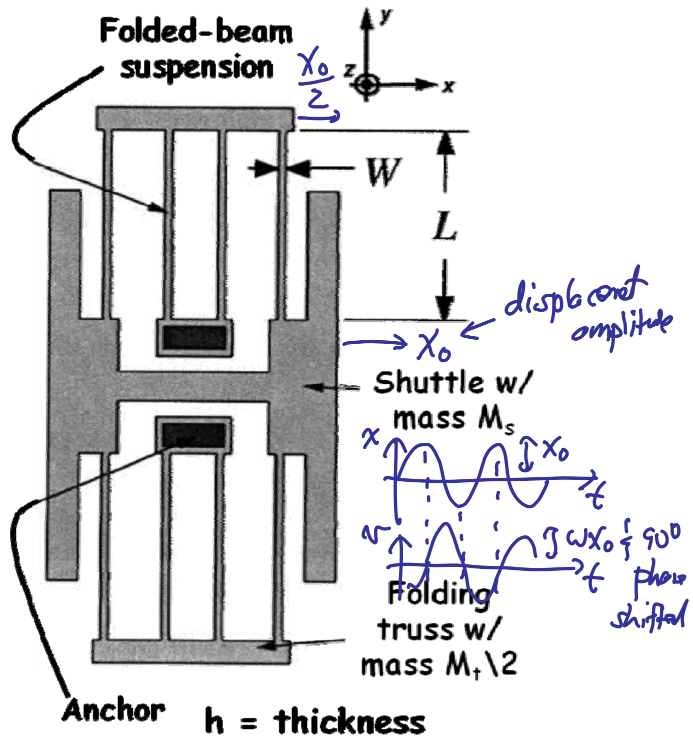
ρ = density of the structural material

W = beam width

h = " thickness

$\hat{y}(x)$ = resonance mode shape

Resonance Freq. of a Folded-Beam Resonator



- Derive an expression for the resonance frequency of the above structure

Use the Rayleigh-Ritz method: (energy method)

$KE_{max} = PE_{max}$

Find the kinetic energy → one piece @ a time!

$$KE_{max} = \underbrace{KE_s}_{\text{shuttle}} + \underbrace{KE_t}_{\text{trusses}} + \underbrace{KE_b}_{\text{beams}}$$

$$KE_{max} = \frac{1}{2} v_s^2 M_s + \frac{1}{2} v_t^2 M_t + \frac{1}{2}\int v_b^2 \, dM_b$$

Velocity of the Shuttle: $v_s = \omega_o x_o$

resonance freq.

peak displacement of structure

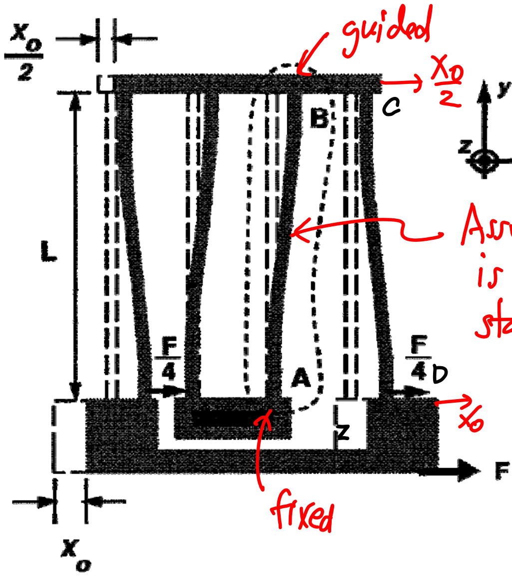
$$\therefore KE_s = \frac{1}{2} v_s^2 M_s = \boxed{\frac{1}{2}\omega_o^2 x_o^2 M_s}$$

Velocity of Truss: $v_t = \frac{1}{2} v_s = \frac{1}{2}\omega_o x_o$

$$\therefore KE_t = \frac{1}{2}\left(\frac{1}{2}\omega_o x_o\right)^2 M_t = \boxed{\frac{1}{8}\omega_o^2 x_o^2 M_t = KE_t}$$

mass of both trusses

Velocity of the Beam Segments:



Assume the mode shape is the same as the static displacement shape

↓

close enough approximation

Segment [AB]:

$$\hat{x}(y) = \frac{F_x}{48 E I_z}\left(3Ly^2 - 2y^3\right), \quad 0 \le y \le L \qquad (1)$$

At $y = L$: $x(L) = \frac{x_o}{2} = \frac{F_x L^3}{48 E I_z}$ ← B.C.

Substitute into (1):

$$\hat{x}(y) = \frac{x_o}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity:

$$v_b(y)\Big|_{[AB]} = \frac{x_o}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_o$$

Plugging into the expression for KE_b:

$$KE_{[AB]} = \frac{1}{2}\int_0^L \frac{x_o^2\omega_o^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

$$= \frac{x_o^2\omega_o^2}{8}\frac{M_{[AB]}}{L}\int_0^L \left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

$M_{[AB]}$: mass (static mass)

mass per unit length

$$\boxed{KE_{[AB]} = \frac{13}{280} x_o^2 \omega_o^2 M_{[AB]}}$$

Segment [CD]:

$$v_b(y)\Big|_{[CD]} = x_o\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]\omega_o$$

Thus:

$$KE_{[CD]} = \frac{X_0^2 \omega_0^2 M_{[CD]}}{2L} \int_0^L \left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]^2 dy$$

$$KE_{[CD]} = \frac{83}{280} X_0^2 \omega_0^2 M_{[CD]}$$

static mass of beam [CD]

Let $M_b \triangleq$ total mass of all 8 beams

Then: $M_{[AB]} = M_{[CD]} = \frac{1}{8} M_b$

Thus:

$$KE_b = 4KE_{[AB]} + 4KE_{[CD]} = \frac{6}{35} X_0^2 \omega_0^2 M_b$$

and

$$KE_{max} = X_0^2 \omega_0^2 \left[\frac{1}{2} M_s + \frac{1}{8} M_t + \frac{6}{35} M_b\right]$$

$PE_{max} \rightarrow$ simply equal to the work done to achieve maximum deflection

$$PE_{max} = \frac{1}{2} k_x X_0^2 = \frac{1}{2} k_c X_0^2$$

Thus, using Rayleigh-Ritz:

$$KE_{max} = PE_{max}$$

$$\cancel{X_0^2} \omega_0^2 \left[\frac{1}{2} M_s + \frac{1}{8} M_t + \frac{6}{35} M_b\right] = \frac{1}{2} k_c \cancel{X_0^2}$$

$$\omega_0 = \sqrt{\frac{k_c}{M_{eq}}}, \text{ where } M_{eq} = M_s + \frac{1}{4} M_t + \frac{12}{35} M_b$$

[Resonance Freq. of a Folded-Beam Suspended Shuttle]

- Go through Module 10 slides 21-31 on your own
- We then went through the project
- Then through the exam solutions
- Then graded exams were passed out