Lecture 14: Beam Bending

- Announcements:
- HW#4 online and due Tuesday, March 18
- Lecture Module 8 online
- Midterm is nearing: Thursday, March 20
  - I will soon pass out materials associated with the midterm, including and information sheet and old exams
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- Reading: Senturia, Chpt. 8
- Lecture Topics:
  - Stress, strain, etc., for isotropic materials
  - Thin films: thermal stress, residual stress, and stress gradients
  - Internal dissipation
  - MEMS material properties and performance metrics
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- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
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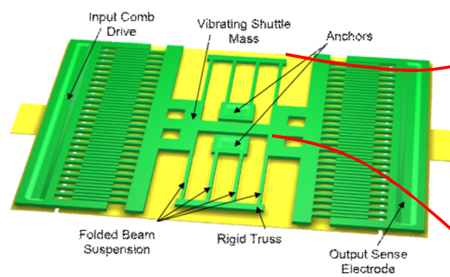
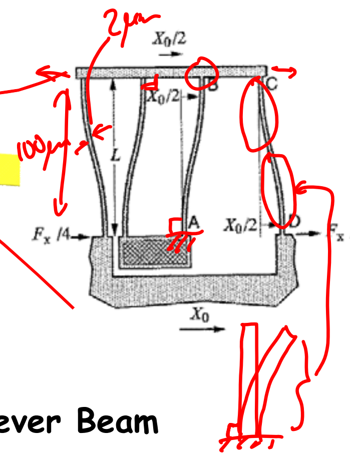
- Last Time:
- Went through Module 7 on Mechanics of Materials
- Now finish this
- Then, start a new topic: Bending of Beams

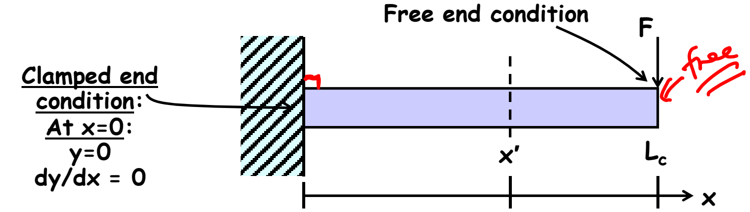
- Springs and suspensions very common in MEMS
- Coils are popular in the macro-world; but not easy to make in the micro-world
- Beams: simpler to fabricate and analyze; become "stronger" on the micro-scale → use beams for MEMS



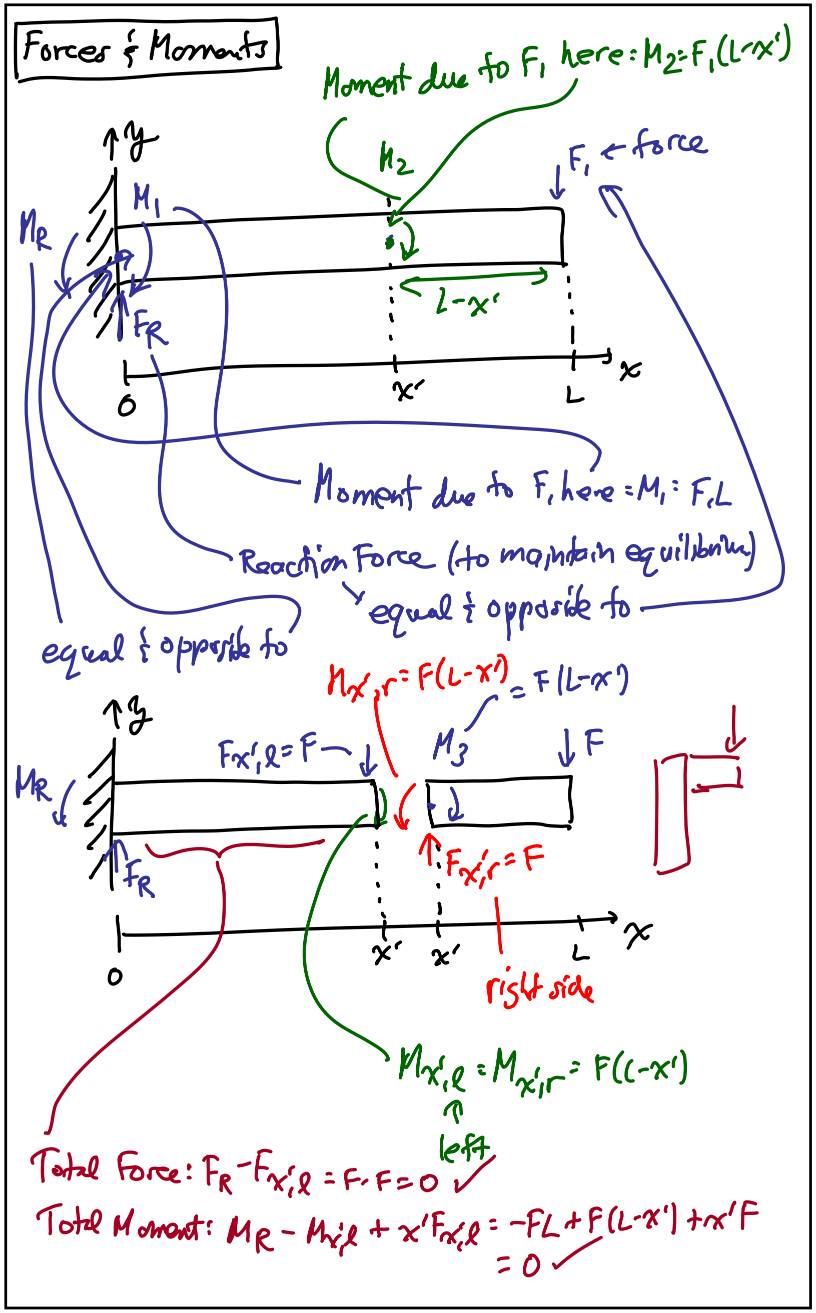


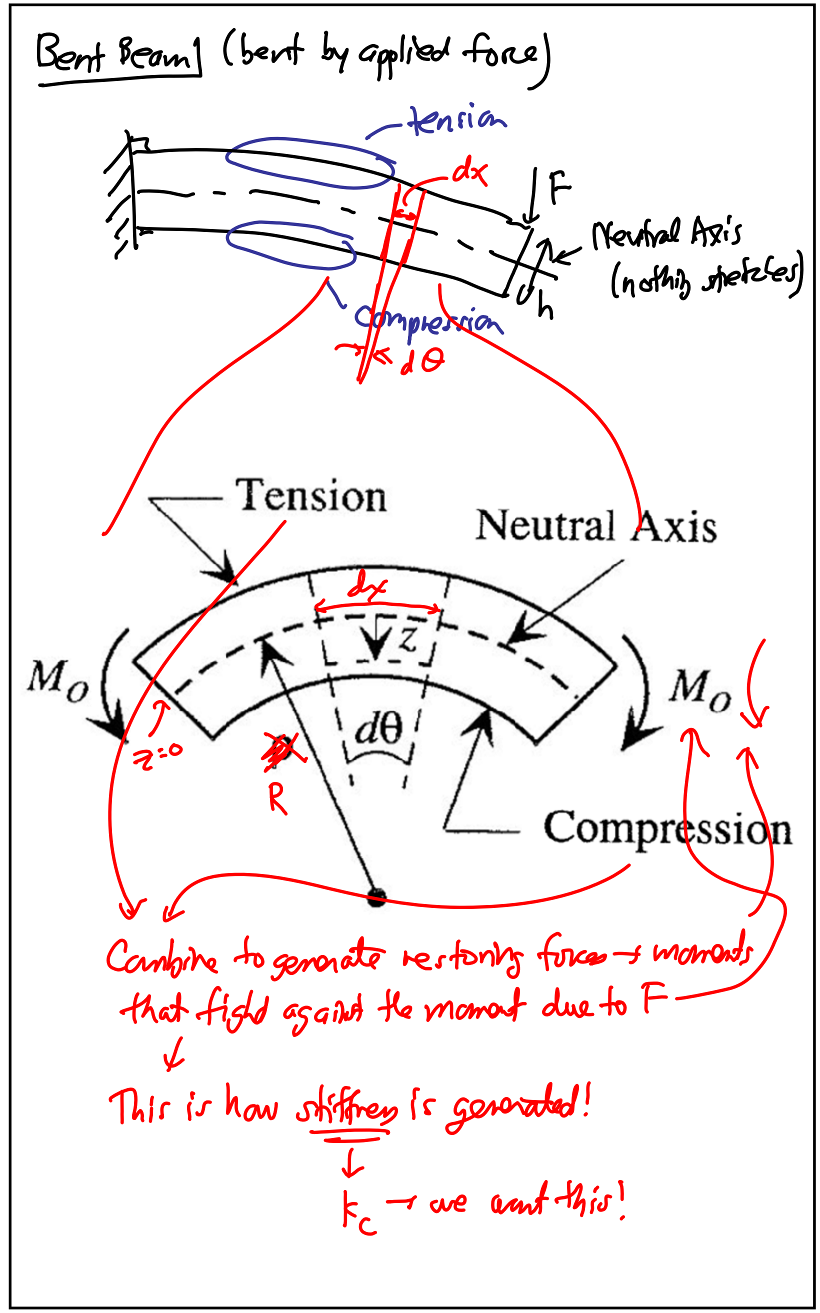
Comb-Driven Folded Beam Actuator

Problem: Bending a Cantilever Beam



- **Objective**: Find relation between tip deflection $y(x=L_c)$ and applied load F → F = ky ultimately, want k
- **Assumptions**:
  1. Tip deflection is small compared with beam length
  2. Plane sections (normal to beam's axis) remain plane and normal during bending, i.e., "pure bending"
  3. Shear stresses are negligible


Forces & Moments
Moment due to F1 here: M2 = F1(L−x')
F1 ← force
M2
M1
MR
FR
L−x'
x' L x 0 y
Moment due to F1 here = M1 = F1L
Reaction Force (to maintain equilibrium)
equal & opposite to
equal & opposite to
Mx',r = F(L−x') = F(L−x')
Fx',l = F
M3
F
Fx',r = F
right side
Mx',l = Mx',r = F(L−x')
left
Total Force: FR − Fx',l = F − F = 0 ✓
Total Moment: MR − Mx',l + x'Fx',l = −FL + F(L−x') + x'F = 0 ✓


Bent Beam (bent by applied force)
tension
dx
F
Neutral Axis (nothing stretches)
h
compression
dθ
Tension
Neutral Axis
dx
z
Mo
Mo
dθ
z=0
R
Compression
Combine to generate restoring forces & moments that fight against the moment due to F
This is how stiffness is generated!
kc → we want this!

Beam Segment in Pure Bending

⇒ consider the segment bounded by the dashed lines defining $d\theta$

At $z=0$: neutral axis → segment length $= dx = R\,d\theta$ (1)

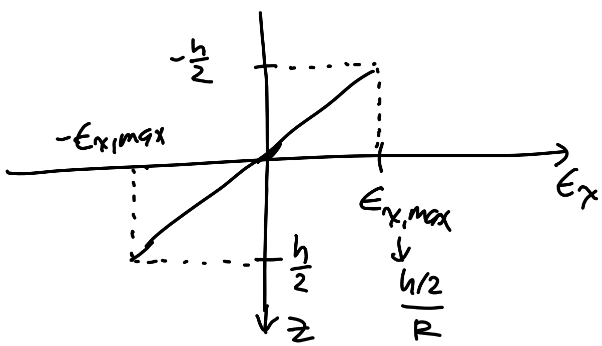
At any z: segment length $= dL = (R-z)\,d\theta$ (2)

Combine (1) & (2): $dL = dx - z\,d\theta = dx - \frac{z}{R}dx$

Thus, the axial strain @ z:

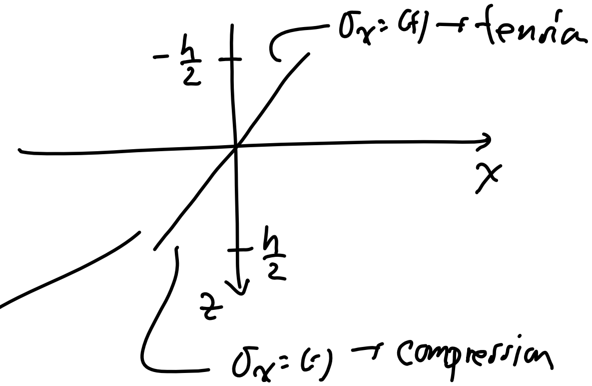
$$\epsilon_x = \frac{dL - dx}{dx} = -\frac{z}{R} \Rightarrow \boxed{\epsilon_x = -\frac{z}{R}}$$

Thus, the strain varies linearly along beam thickness:



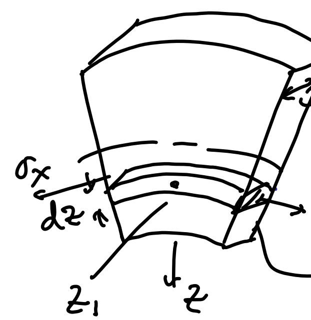
Of course, there is a corresponding axial stress:

$$\sigma_x = \epsilon_x E = \boxed{-\frac{zE}{R} = \sigma_x}$$



This gradient of stress generates a bending moment (in response to original applied normal force → moment)

Stress → Force:



$\sigma_{x,z_1} = \frac{F}{A} \rightarrow F = (w \cdot dz)\cdot\sigma_{x,z_1}$

$A = w\,dz$

⇒ integrate stress through the thickness of the beam

differential ~~thing~~ moments

moment arm

$$M = \int_{-\frac{h}{2}}^{\frac{h}{2}} [\underbrace{(Wdz)\sigma_x}_{\text{force}}]\cdot z$$

$$= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{EWz^2}{R} dz \Rightarrow \boxed{M = -\left(\frac{1}{12}Wh^3\right)\frac{E}{R}}$$

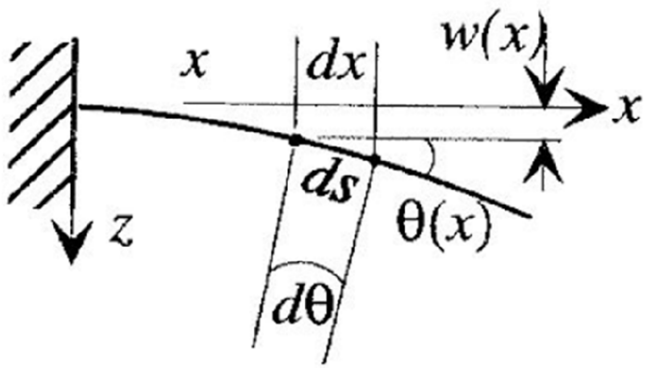
$$\left(\sigma_x = -\frac{zE}{R}\right)$$

$\frac{1}{12}Wh^3 = I \triangleq$ Moment of Inertia

$$\boxed{\frac{1}{R} = -\frac{M}{EI}}$$

Note: (+) radius of curvature ↓ (−) internal bending moment

Differential Equation for Beam Bending



Write out some geometric relationships:

⇒ then use small angle approx:

$$\cos\theta = \frac{dx}{ds} \rightarrow ds = \frac{dx}{\cos\theta} \longrightarrow ds \cong dx$$

$$\tan\theta = \frac{dw}{dx} = \text{slope of the beam at the point of interest} \longrightarrow \theta \cong \frac{dw}{dx} \quad (1)$$

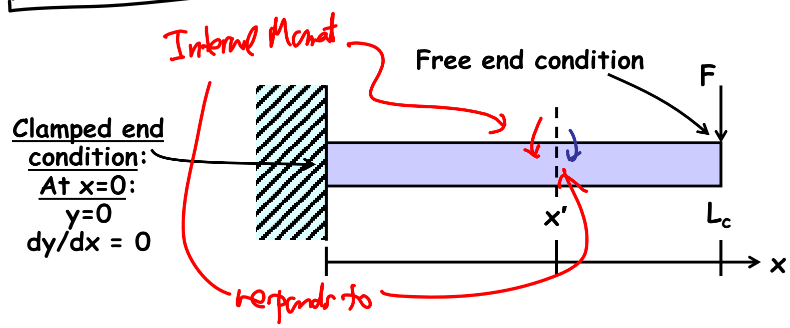
$$ds = Rd\theta \rightarrow \frac{1}{R} = \frac{d\theta}{ds} \longrightarrow \frac{1}{R} = \frac{d\theta}{dx} \quad (2)$$

Inserting (1) into (2):

$$\frac{1}{R} = \boxed{\frac{d^2w}{dx^2} = -\frac{M}{EI}}$$

Diff. Eq. for Small Angle Beam Bending

$\boxed{\text{Cantilever Beam w/ Concentrated Load}}$



Internal Moment @ position x: $M = -F(L-x)$

Thur:

$$\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

w/ { Clamped End B.C.: $w(x=0)=0$, $\frac{dw}{dx}(x=0)=0$
Free End:

Solve to get w:

⇒ use Laplace; or a trivial solution:

$w = A + Bx + Cx^2 + Dx^3$, then apply B.C.'s

$$\boxed{w = \frac{FL}{2EI}x^2\left(1-\frac{x}{3L}\right)}$$

Deflection @ x due to a point load F applied @ $x=L$.

Maximum Deflection → occurs $x=L$:

$$w_{max} = \left(\frac{L^3}{3EI}\right)F \rightarrow F = \underbrace{\left(\frac{3EI}{L^3}\right)w(x=L)}_{= k_c w(x=L)}$$

stiffness @ $x=L$

where $\boxed{k_c = \frac{3EI}{L^3}}$

$\left[I = \frac{1}{12}wh^3\right] \Rightarrow \boxed{k_c = \frac{1}{4}Ew\frac{h^3}{L^3}}$

Ex: $L=100\mu m$, $W=2\mu m$, $h=2\mu m$

polysilicon → $E = 150$ GPa

$$k_c = \frac{1}{4}(150G)(2\mu)\left(\frac{2\mu}{100\mu}\right)^3 = 0.6\ N/m$$

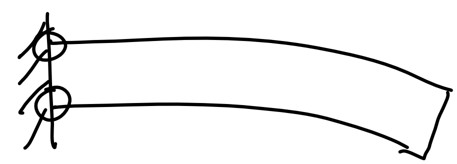
Maximum Stress in a Bent Cantilever

From before, the radius of curvature is given by

$$\frac{1}{R} = \frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

⇒ $\frac{1}{R}$ is maximized (i.e., R is minimized) when $x=0$

$[x=0]$: $\frac{1}{R} = \frac{d^2w}{dx^2} = \frac{FL}{EI}$



Strain is maximized:

1. At the top surface → tensile
2. At the bottom surface → compressive

$$\epsilon_{max} = \frac{z}{R} = \frac{h}{2}\frac{1}{R} = \boxed{\frac{h}{2}\frac{FL}{EI} = \epsilon_{max}}$$