Lecture 15: Stress Gradients

- **Announcements:**
- HW#4 online and due Tuesday, March 18
- Lecture Module 8 online
- Midterm is nearing: Thursday, March 20
  - I will soon pass out materials associated with the midterm (info sheet and old exams)
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- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
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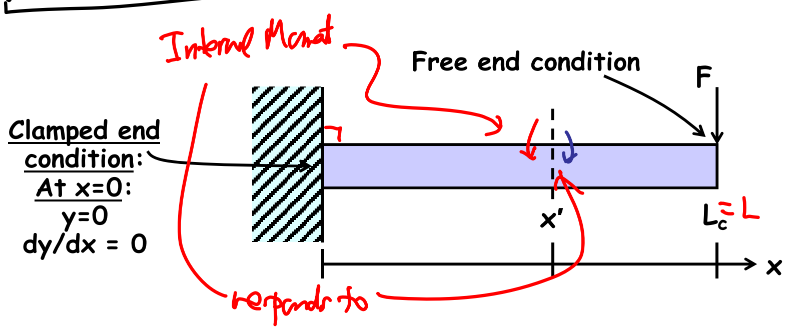
- **Last Time:**

$\boxed{\text{Cantilever Beam w/ Concentrated Load}}$



Internal Moment @ position x: $M = -F(L-x)$

Thus:

$$\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

w/ $\begin{cases} \text{Clamped End B.C.: } w(x=0)=0,\ \frac{dw}{dx}(x=0)=0 \\ \text{Free End:} \end{cases}$

Solve to get w:

→ use Laplace; or a trivial solution:

$w = A + Bx + Cx^2 + Dx^3$, then apply B.C.'s

$$\boxed{w = \frac{FL}{2EI}x^2\left(1-\frac{x}{3L}\right)}$$

Deflection @ x due to a point load F applied @ $x=L$.

Maximum Deflection → occurs $x=L$:

$$w_{max} = \left(\frac{L^3}{3EI}\right)F \rightarrow F = \underbrace{\left(\frac{3EI}{L^3}\right)w(x=L)}_{= k_c w(x=L)}$$

stiffness @ $x=L$

where $\boxed{k_c = \frac{3EI}{L^3}}$

$\left[I = \frac{1}{12}wh^3\right] \Rightarrow \boxed{k_c = \frac{1}{4}Ew\frac{h^3}{L^3}}$

Ex: $L = 100\,\mu m$, $W = 2\,\mu m$, $h = 2\,\mu m$

polysilicon → $E = 150\ GPa$

$$k_c = \frac{1}{4}(150G)(2\mu)\left(\frac{2\mu}{100\mu}\right)^3 = \underline{0.6\ N/m}$$

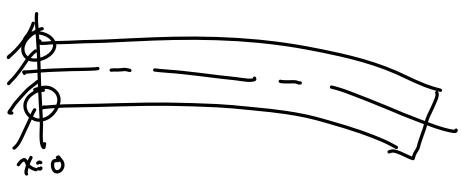
Maximum Stress in a Bent Cantilever

From before, the radius of curvature is given by

$$\frac{1}{R} = \frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

$\Rightarrow \frac{1}{R}$ is maximized (i.e., R is minimized) when $x=0$

$$[x=0] \Rightarrow \frac{1}{R} = \frac{d^2w}{dx^2} = \frac{FL}{EI}$$



Strain is maximized:

1. At the top surface → tensile
2. At the bottom surface → compressive

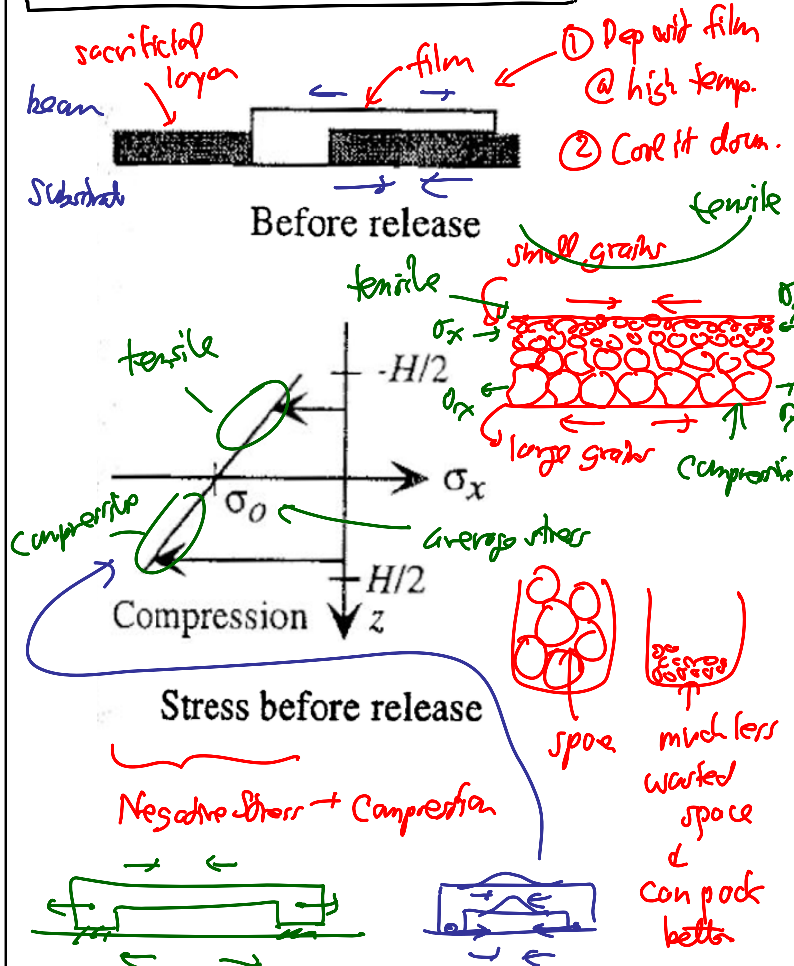
$$\epsilon_{max} = \frac{z}{R} = \frac{h}{2}\frac{1}{R} = \boxed{\frac{h}{2}\frac{FL}{EI} = \epsilon_{max}}$$

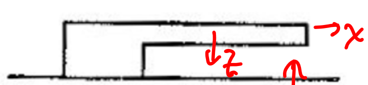
$$\left[I = \frac{1}{12}Wh^3\right] \Rightarrow \epsilon_{max} = \frac{\cancel{h}}{2}\frac{FL}{E}\left(\frac{12}{Wh^{\cancel{3}\,2}}\right) = \boxed{\frac{6L}{EWh^2}F = \epsilon_{max}}$$

$$\boxed{\sigma_{max} = \epsilon_{max}E = \frac{6L}{Wh^2}F}$$

(maximum stress in a bent cantilever subjected to a force F at its tip)

Stress (Strain) Gradients in Cantilevers

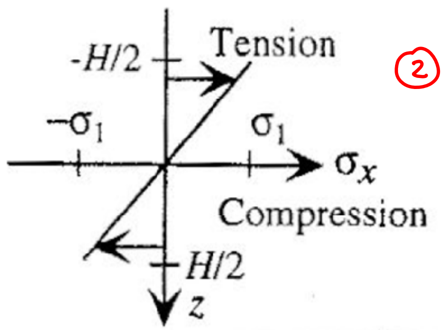




} Intermediate State

After release,
but before bending

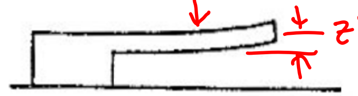
① Removed the sac. layer
↳ beam free to do what it wants



② beam stretches → removes average axial stress
↓
③ Bends to relieve stress gradient

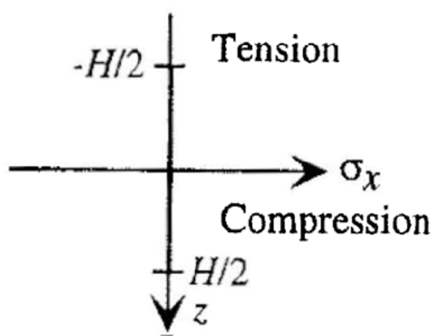
Stress after release,
but before bending

average axial stress = 0



After bending

All stress relieved
↓
state of minimum energy



After bending

Bending Due to Stress Gradient

Find the radius of curvature.

Prior to release, total axial stress: $\sigma = \sigma_0 - \frac{\sigma_1}{(H/2)} z$

The internal moment:

$$M_x = \int_{-\frac{H}{2}}^{\frac{H}{2}} [(W dz)\cdot\sigma]\cdot z = \int_{-\frac{H}{2}}^{\frac{H}{2}} W\left(z\sigma_0 - \frac{\sigma_1 z^2}{(H/2)}\right)dz$$

$$= W\left(\frac{1}{2}\sigma_0 z^2 - \frac{2\sigma_1 z^3}{3H}\right)\Bigg|_{-\frac{H}{2}}^{\frac{H}{2}}$$

$$= W\left(\cancel{\frac{1}{2}\sigma_0\frac{H^2}{4}} - \frac{2}{3}\sigma_1\frac{H^2}{8} - \cancel{\frac{1}{2}\sigma_0\frac{H^2}{4}} - \frac{2}{3}\sigma_1\frac{H^2}{8}\right)$$

average stress cancels out

$$\boxed{M_x = -\frac{1}{6}\sigma_1 W H^2}$$

Thus, the radius of curvature:

$$\frac{1}{R} = -\frac{M_x}{E'I} \rightarrow R = \frac{E'I}{M_x} = \frac{1}{2}\frac{E'H}{\sigma_1}$$

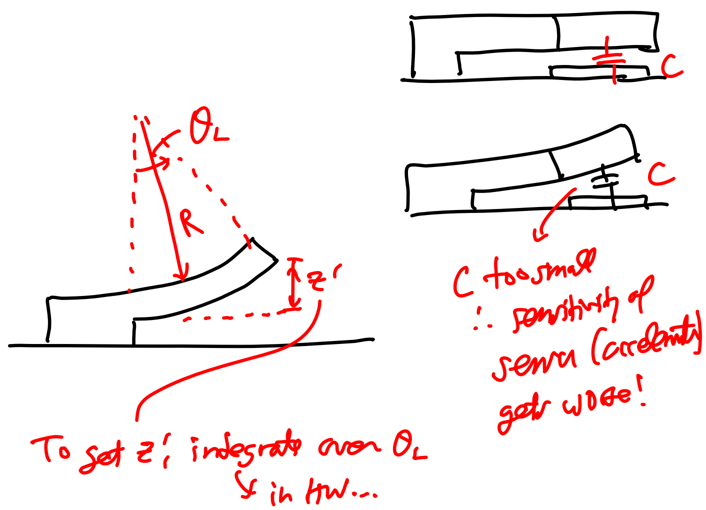
$$\left[I = \frac{1}{12}WH^3\right]$$

Biaxial Modulus

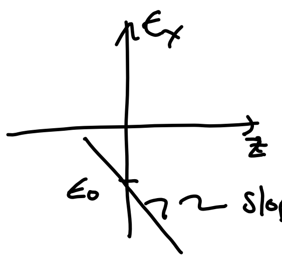
$$\boxed{R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1}}$$

[Radius of Curvature for a Cantilever w/ a Stress Gradient]

Radius of Curvature → z's



Definition: Strain Gradient

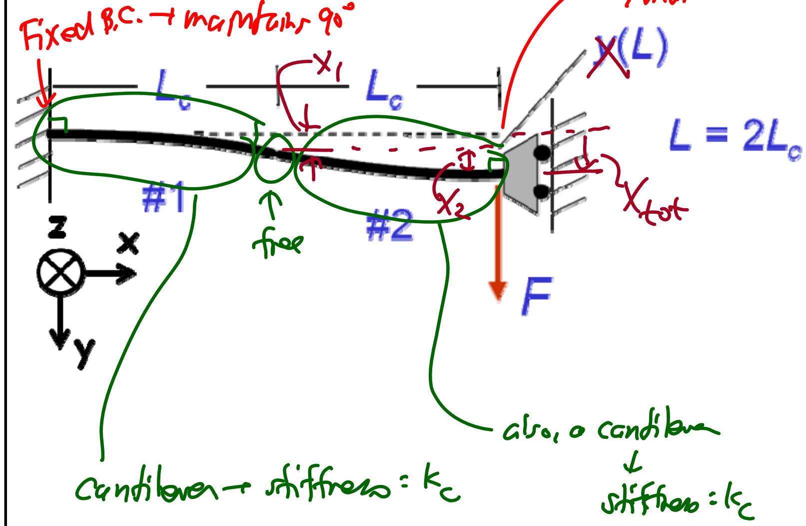


slope $\triangleq$ Strain Gradient $= \Gamma$

$$\Gamma = \frac{\epsilon_1}{(H/2)}$$

$$R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1} = \frac{H}{2}\frac{E'}{\sigma_1} = \frac{(H/2)}{\epsilon_1} = \frac{1}{\Gamma} \rightarrow \boxed{\Gamma = \frac{1}{R}}$$

Series Combination of Springs



Series: $x_{tot} > x_1$, $x_{tot} > x_2$ $\}$ $x_{tot} = x_1 + x_2$