Lecture 17: Beam Combos II

- Announcements:
- Midterm is nearing: Thursday, March 20
  - I passed out materials associated with the midterm (info sheet and old exams) last lecture
- ------------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- ------------------------------------
- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
  - Estimating Resonance Frequency
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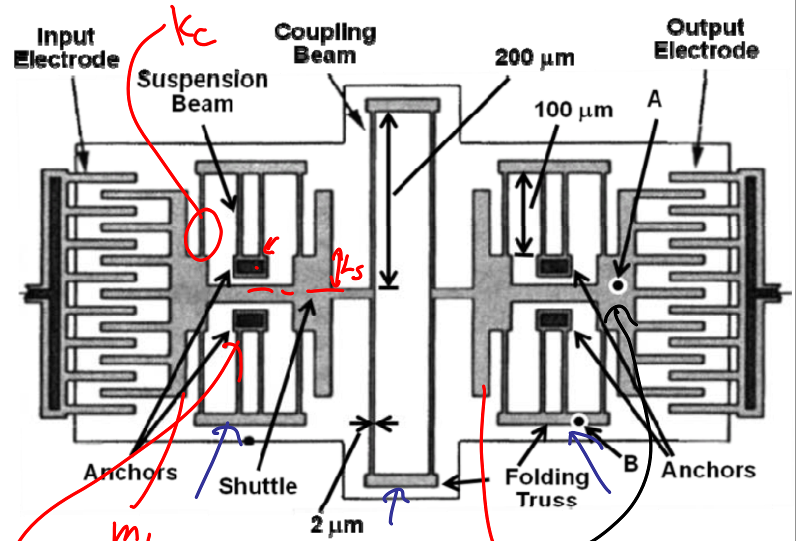
- Last Time:
- Spring constant determination
- Going through examples

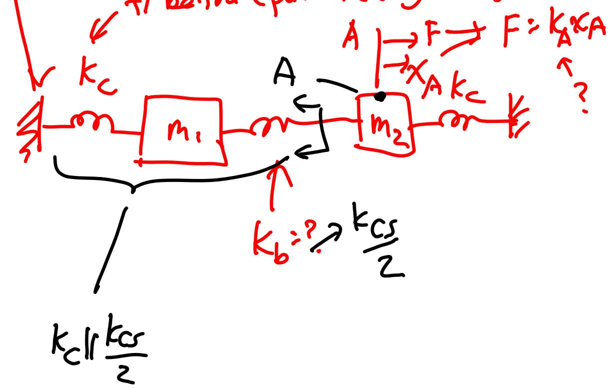
over

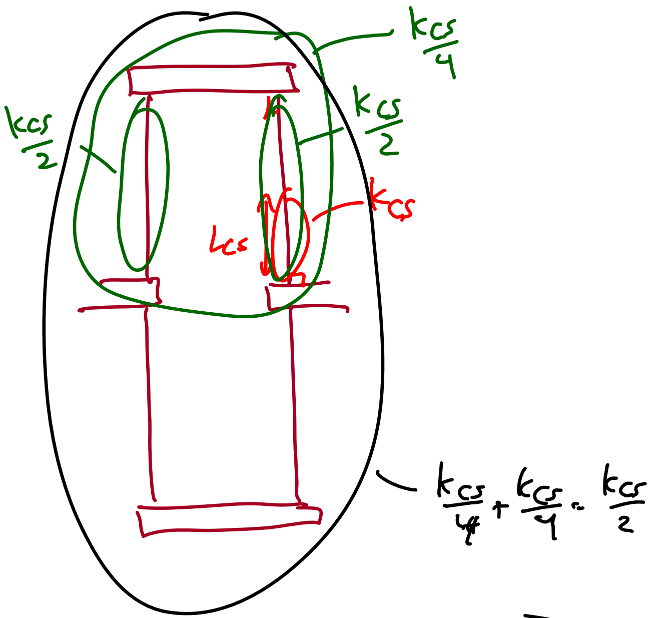
Micromechanical Filter Example



⇒ Find the stiffness @ pt. A

(shuttles & trusses are rigid)





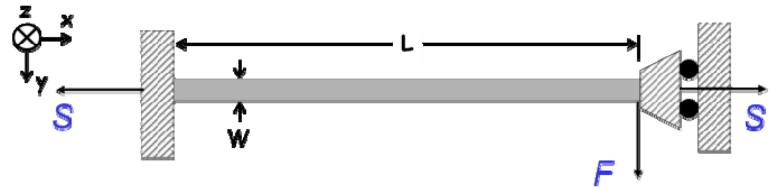
$$\frac{k_{cs}}{4}+\frac{k_{cs}}{4}=\frac{k_{cs}}{2}$$

$\therefore\ k_A = k_c + k_c \,\|\, \frac{k_{cs}}{2}$ where $k_c = \frac{24EI_z}{L_b^3}$

$k_{cs} = \frac{24EI_z}{L_{cs}^3}$

Tensioned Spring (Non-Ideality)

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$



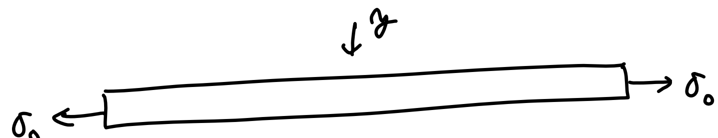
Governing differential equation: (Euler Beam Equation)

$$EI_z\frac{d^4y}{dx^4} - S\frac{d^2y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

Axial Load

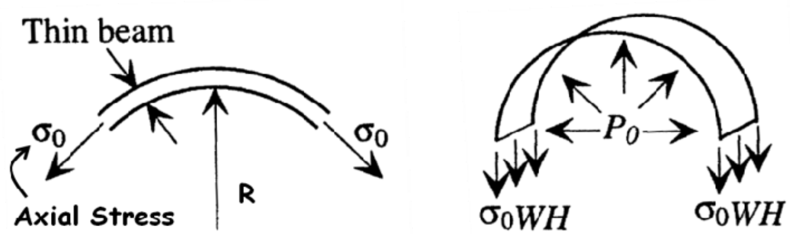
Heuristic Derivation for the Euler Beam Equation

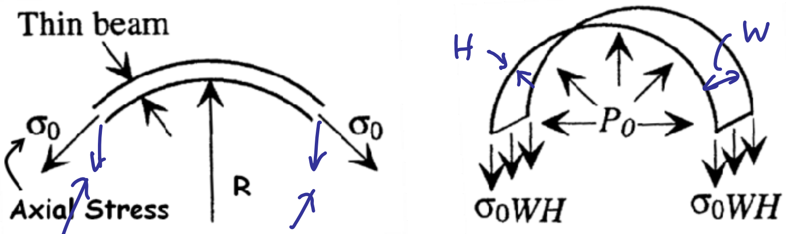
Consider first a straight beam under an axial stress:



⇒ no effect on y-directed stiffness when the beam is straight

...but, when the beam is bent:



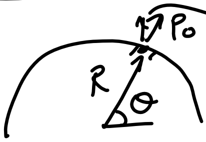


y-directed component
↳ now, k is affected!

Need upward pressure P_0 to counteract the downward force from to keep everything in static equilibrium

For ease of analysis:
Assume the beam is bent to an angle π
↳ downward vertical force : $2\sigma_0 WH$

Upward force due to P_0:



$P_y(\theta) = P_0 \sin\theta$

$$F_u = \int_0^{\pi} (P_0 \sin\theta) W (R d\theta)$$

$$= -P_0 W R \cos\theta \Big|_0^{\pi}$$

$$F_u = 2RWP_0$$

[Equilibrium] ⇒ $2RWP_0 = 2\sigma_0 WH \rightarrow P_0 = \frac{\sigma_0 H}{R}$

$\left[q_0 = \frac{\text{beam load}}{\text{unit length}} = P_0 W,\ \frac{1}{R} = \frac{d^2 w}{dx^2}\right]$ → beam displacement in y-direction

$q_0 = \sigma_0 W H \frac{d^2 w}{dx^2}$ ← generalize for small displacements & angles

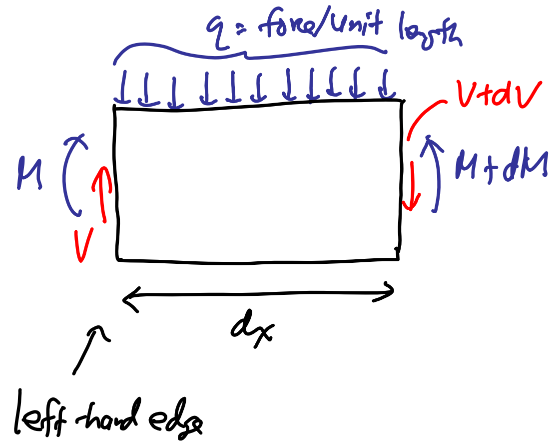
Use the differential beam bending equation:

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \longrightarrow \frac{d^4 w}{dx^4} = \frac{q}{EI}$$

↑ ???

← load/unit length

Relationships Between Forces on a Fully Loaded Differential Beam Element



[Total Static Equilibrium] ⇒ total force = 0

F_T: total force: $q\,dx + (\cancel{V}+dV) - \cancel{V} = 0$

$$\therefore \boxed{\frac{dV}{dx} = -q} \qquad (1)$$

⇒ also, total moment w.r.t. left-hand edge = 0

$$M_T: (\cancel{M}+dM) - \cancel{M} + \cancel{\frac{1}{2}q\,dx^2} - (V+\cancel{dV})\,dx = 0$$

neglect products of differentials

$$\int_0^{dx}(q\,du)\,u = \frac{1}{2}q\,dx^2$$

$$dM - V\,dx = 0 \to \boxed{\frac{dM}{dx} = V} \qquad (2)$$

Using (1) & (2):

$$\left[\frac{d^2M}{dx^2} = \frac{dV}{dx} = -q\right]$$

$$EI\frac{d^4w}{dx^4} = q + q_0$$

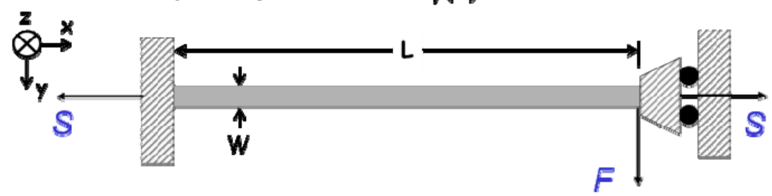
external load

equiv. load from axial stress

$$\left[q_0 = \sigma_0 W H \frac{d^2w}{dx^2}\right] \Rightarrow$$

$$EI\frac{d^4w}{dx^4} - \underbrace{(\sigma_0 W H)}_{\text{tension force in the beam} = S}\frac{d^2w}{dx^2} = q$$

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$



Governing differential equation: (Euler Beam Equation)

$$EI_z\frac{d^4y}{dx^4} - \underbrace{S}_{\text{Axial Load}}\frac{d^2y}{dx^2} = F\underbrace{\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

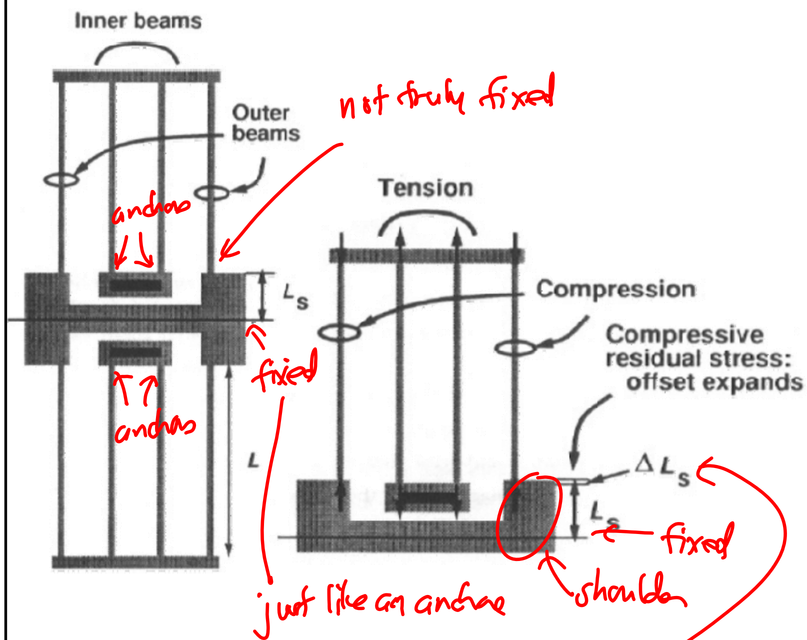
- Can solve the ODE using standard methods
  - Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)
  - For solution to the clamped-guided case: see S. Timoshenko, Strength of Materials II: Advanced Theory and Problems, McGraw-Hill, New York, 3rd Ed., 1955
- Result from Timoshenko:

$S > 0$ (tension) $\quad k^{-1} = \dfrac{pL - 2\tanh(pL/2)}{p|S|} = \dfrac{y(x=L)}{F}$

$S < 0$ (compression)

$$k^{-1} = \frac{-pL + 2\tan(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

where $p = \sqrt{\dfrac{|S|}{EI_z}}$



this generates the strain

① If polySi strain is ϵ_r, then the shoulder expands by $\Delta L_s = \epsilon_r L_s$

② This then applies a load to the beams $\rightarrow \Delta L = \Delta L_s$

③ Beam Stress:

$$\epsilon_b = \frac{\Delta L}{2L} = \frac{\Delta L_s}{2L} = \pm \epsilon_r \frac{L_s}{2L}$$

Stress Force: $\boxed{S = \pm E \epsilon_r \left(\frac{L_s}{2L}\right) Wh}$ (axial tension)

④ Spring Constants:

$$k = 4\underbrace{(k_{com}^{-1} + k_{ten}^{-1})^{-1}}_{k_{com} \parallel k_{ten}}$$

$$k = 4\left[\frac{-pL + 2\tan(pL/2)}{pS} + \frac{pL - 2\tanh(pL/2)}{p|S|}\right]^{-1}$$