Lecture 19: Resonance Frequency

- **Announcements**:
- HW#5 online
- Module 10 online
- Pass out project today (in middle of class)
  - Project description
  - Check point deadlines
  - Find two partners ASAP – you'll work in groups of three
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- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
- ------------------------------------
- **Last Time**:
- Passed out graded midterms & solutions
- Started into Module 10

$\boxed{\text{Estimating Resonance Frequency}}$

$\longmapsto x(t) = X_0 \cos\omega t$

Potential Energy

$$W(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kX_0^2\cos^2\omega t$$

Kinetic Energy

$$K(t) = \frac{1}{2}m\dot{x}^2(t) = \frac{1}{2}mX_0^2\underbrace{\omega^2\sin^2\omega t}$$

$\dot{x} = \frac{dx}{dt}$: velocity

Remarks:

1. Energy must be conserved.
2. Total Energy = Potential Energy + Kinetic Energy at all times & locations of the structure

$$W_{max} = \frac{1}{2}kX_0^2 = K_{max} = \frac{1}{2}m\omega_0^2X_0^2$$

max potential energy → when $x = X_0$

peak displacement

max kinetic energy → when $x = 0$

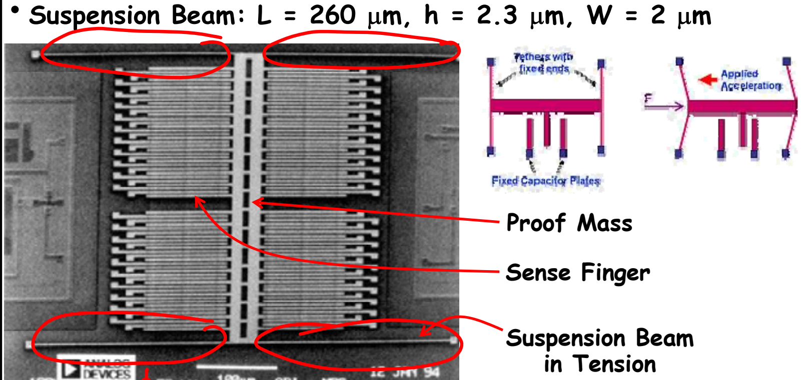
$$\boxed{\omega_o = \sqrt{\frac{k}{m}}}$$

radian resonance frequency

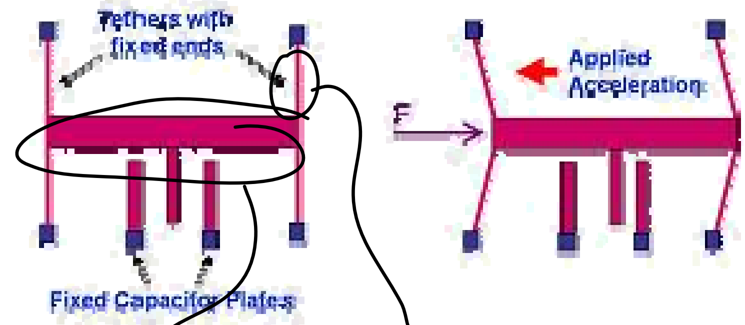
→ good for problems where mass & stiffness can be separated; i.e., they are distinct

Ex: ADXL-50

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm



In fabrication, purposely introduce tensile stress into the support beams: sizable

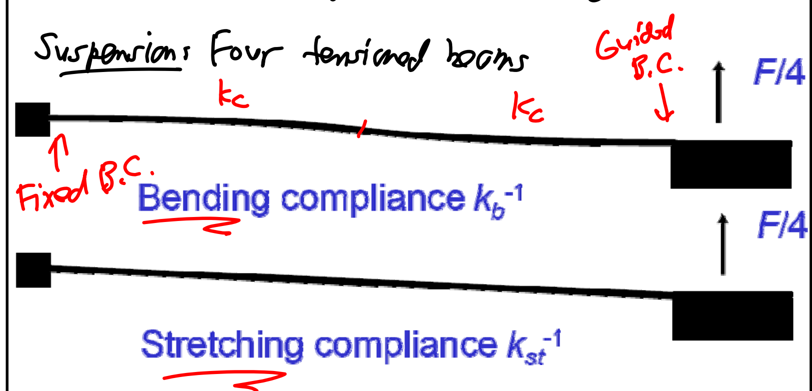


Mass of structure >> mass of springs
∴ ignore the mass of the springs

Stiffness of springs << stiffness of structure
∴ ignore the stiffness of the structure

For the ADXL-50: 60% of the mass comes from the sense fingers → M = 162 ng

Suspension: Four tensioned beams

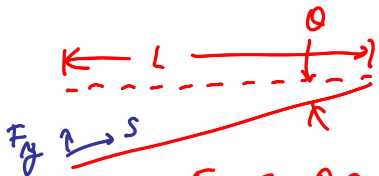


Bending Contribution

$$k_b^{-1} = \left(\frac{1}{k_c}+\frac{1}{k_c}\right) = 2\left(\frac{(L/2)^3}{3E(hw^3/12)}\right) = \frac{L^3}{Ehw^3}$$

$$= 4.2\mu m/\mu N$$

Stretching Contribution



$\approx \tan\theta$

$$F_y = S\sin\theta \approx S\theta \approx S\left(\frac{y}{L}\right) = \underbrace{\left(\frac{S}{L}\right)}_{k_{st}} y$$

[assume small displacements]

stretching stiffness

$$k_{st}^{-1} = \frac{L}{S} = \frac{L}{\sigma_r W h} = 1.14\mu m/\mu N$$

To get the total spring constant

→ add the bending stiffness to the stretching:

$$k = 4(k_b + k_{st}) = 4(0.24 + 0.88) = 4.5\mu N/\mu m$$

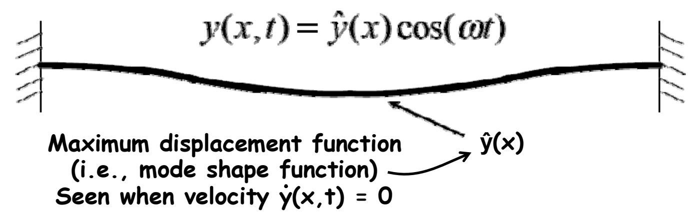
Now, get resonance freq:

$$f = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \frac{1}{2\pi}\sqrt{\frac{4.48\ N/m}{162\times10^{-12}kg}} = 26.5kHz$$

ADXL-50 Data Sheet: $f_o = 24kHz$ ← difference?

capacitive transducer → electrical stiffness

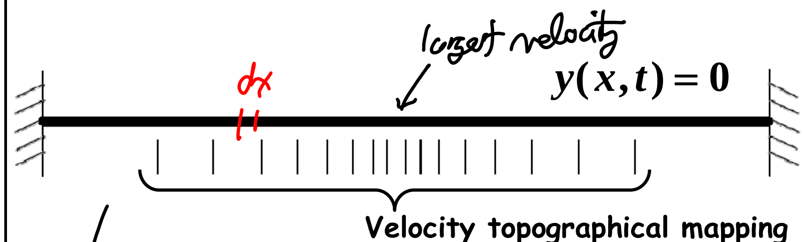
- Vibrating structure displacement function:



- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy W_{max} at the point of maximum displacement (e.g., when t=0, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

Get Maximum Kinetic Energy

velocity: $v(x) = \frac{\partial y(x,t)}{\partial t} = -\omega\hat{y}(x)\sin\omega t$



When $y(x,t) = 0$, all the energy in the structure is kinetic ($W = 0$)

↓

$$v\left(x, \frac{(2m+1)\pi}{2\omega}\right) = -\omega\hat{y}(x)$$

$$t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \ldots$$

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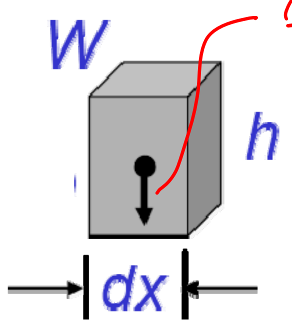


velocity

$v = -\omega \hat{y}(x)$ $\qquad \frac{(2m+1)\pi}{2w}$

$dk = \frac{1}{2} \cdot dm \cdot [v(x,t)]^2$

$dm = \rho(Wh\,dx)$

density

Maximum K.E.:

$K_{max} = \int_0^L \frac{1}{2} \rho W h\,dx\, v^2(x,t) = \int_0^L \frac{1}{2} \rho W h \omega^2 [\hat{y}(x)]^2 dx$

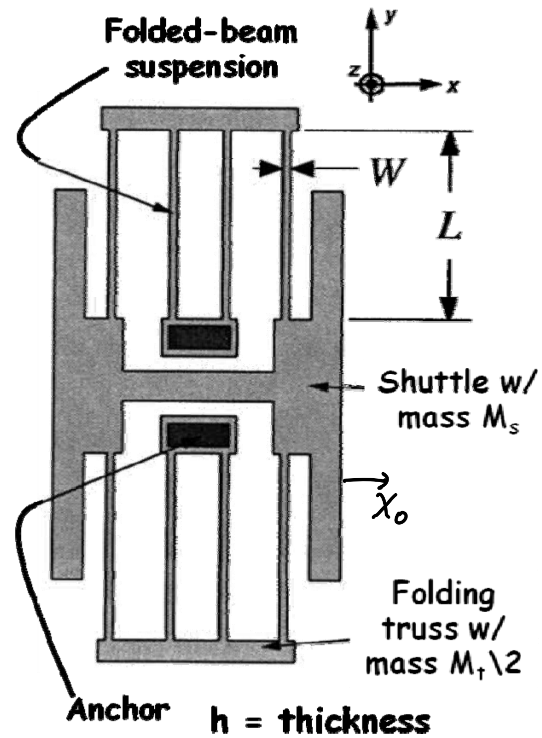
To get frequency:

$K_{max} = W_{max}$

$$\therefore \omega = \sqrt{\frac{W_{max}}{\int_0^L \frac{1}{2} \rho W h [\hat{y}(x)]^2 dx}}$$

ω: radian resonance freq. W = beam width
W_{max}: max. potential energy h = " thickness
ρ: density of the structural material
$\hat{y}(x)$: resonance mode shape

Resonance Freq. of a Folded-Beam Structure



- Derive an expression for the resonance frequency of the above structure

Use the Rayleigh-Ritz Method: (energy method)

$KE_{max} = PE_{max}$

Find the kinetic energy → one piece at a time

$KE_{max} = \underbrace{KE_s}_{\text{shuttle}} + \underbrace{KE_t}_{\text{truss}} + \underbrace{KE_b}_{\text{beams}}$

$$KE_{max} = \frac{1}{2} v_s^2 M_s + \frac{1}{2} v_t^2 M_t + \frac{1}{2} \int v_b^2 \, dM_b$$

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