Lecture 20: Resonance Frequency & Equiv Ckts. I

- Announcements:
- Project handed out and described last time
- You should be partnered up, 3 per team
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- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
- Reading: Senturia, Chpt. 5
- Lecture Topics:
  - Lumped Mechanical Equivalent Circuits
  - Electromechanical Analogies
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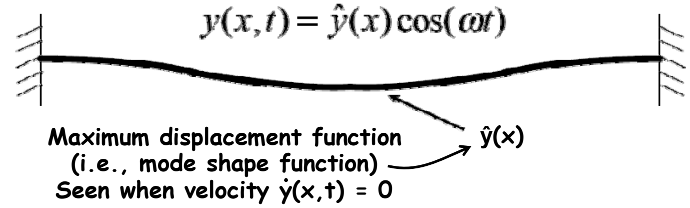
- Last Time:
- Started resonance frequency determination

over

- Vibrating structure displacement function:

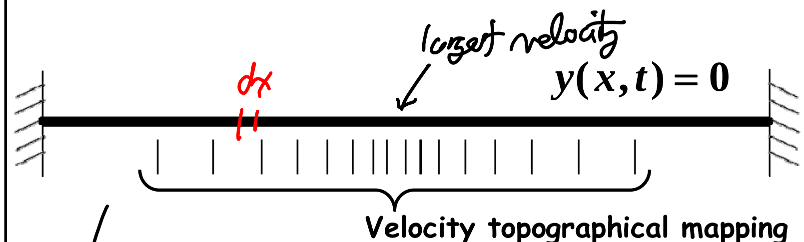
$$y(x,t) = \hat{y}(x)\cos(\omega t)$$



- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy W_{max} at the point of maximum displacement (e.g., when $t=0$, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

Get Maximum Kinetic Energy

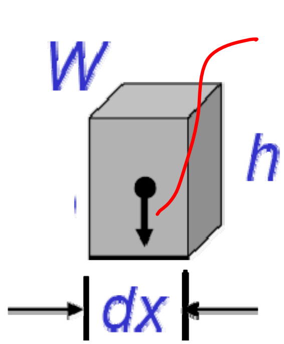
velocity: $v(x) = \frac{\partial y(x,t)}{\partial t} = -\omega\hat{y}(x)\sin\omega t$



When $y(x,t)=0$, all the energy in the structure is kinetic ($W=0$)

$$v\left(x, \frac{(2m+1)\pi}{2\omega}\right) = -\omega\hat{y}(x)$$

$t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \ldots$



velocity

$$v = -\omega \hat{y}(x)$$

$$\frac{(2m+1)\pi}{2W}$$

$$dk = \frac{1}{2} \cdot dm \cdot [v(x,t)]^2$$

$$dm = \rho(Wh\,dx)$$

density

Maximum K.E.:

$$K_{max} = \int_0^L \frac{1}{2}\rho W h\,dx\, v^2(x,t) = \int_0^L \frac{1}{2}\rho W h \omega^2 [\hat{y}(x)]^2 dx$$

To get frequency:

$$K_{max} = W_{max}$$

$$\therefore \boxed{\omega = \sqrt{\frac{W_{max}}{\int_0^L \frac{1}{2}\rho W h [\hat{y}(x)] dx}}}$$

ω: radian resonance freq.

W_{max}: max. potential energy

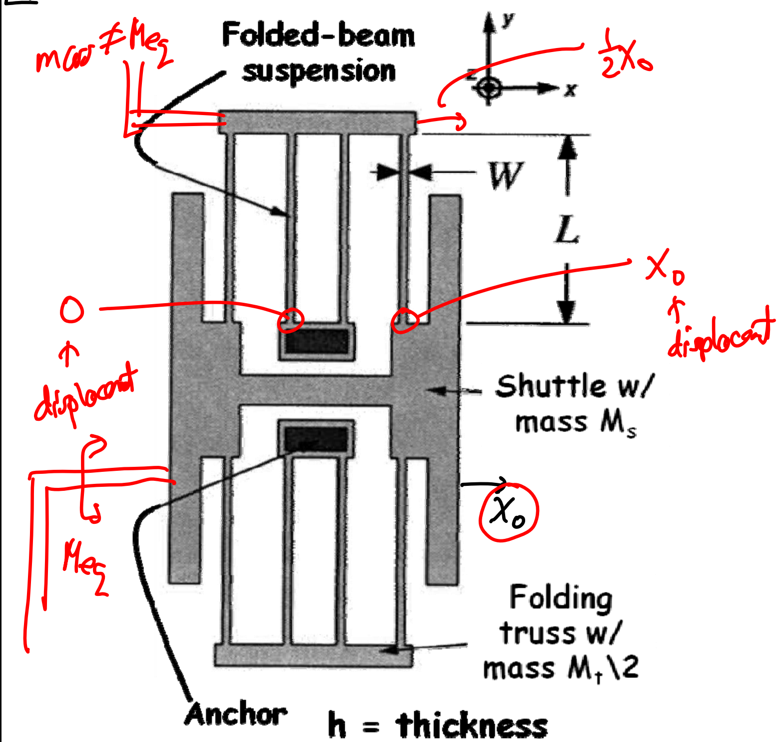
ρ: density of the structural material

$\hat{y}(x)$: resonance mode shape

W = beam width

h = " thickness

Resonance Freq. of a Folded-Beam Structure



- Derive an expression for the resonance frequency of the above structure

Use the Rayleigh-Ritz Method: (energy method)

$$KE_{max} = PE_{max}$$

Find the kinetic energy → one piece at a time

$$KE_{max} = \underbrace{KE_s}_{\text{shuttle}} + \underbrace{KE_t}_{\text{truss}} + \underbrace{KE_b}_{\text{beams}}$$

$$KE_{max} = \frac{1}{2}v_s^2 M_s + \frac{1}{2}v_t^2 M_t + \frac{1}{2}\int v_b^2\, dM_b$$

Velocity of the Shuttle: $v_s = \omega_0 X_0$

resonance freq. — maximum displacement of the shuttle

ρV = static mass

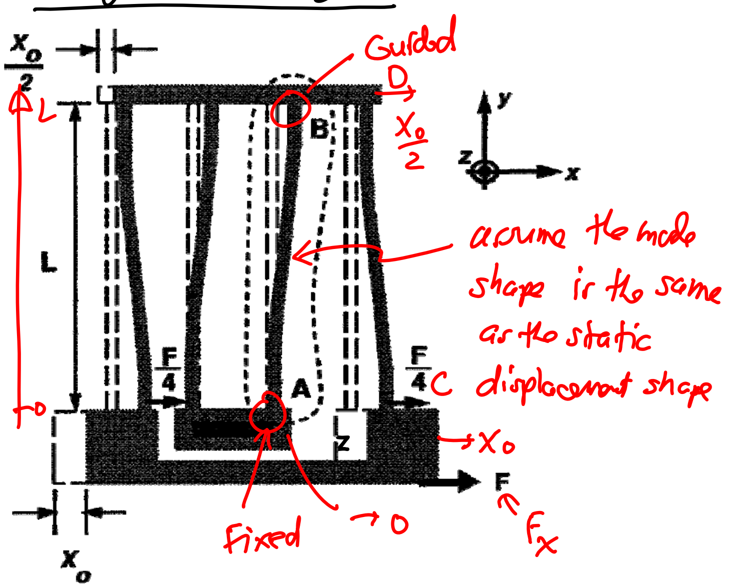
$$\therefore KE_s = \frac{1}{2}v_s^2 M_s = \frac{1}{2}\omega_0^2 X_0^2 M_s$$

Velocity of Truss: $v_t = \frac{1}{2}v_s = \frac{1}{2}\omega_0 X_0$

$$\therefore KE_t = \frac{1}{2}\left(\frac{1}{2}\omega_0 X_0\right)^2 M_t = \frac{1}{8}\omega_0^2 X_0^2 M_t = KE_t$$

static mass for both trusses

Velocity of the Beam Segments:



Segment [AB]:

$$\hat{x}(y) = \frac{F_x}{48EI_z}\left(3Ly^2 - 2y^3\right), \quad 0 \le y \le L \qquad (1)$$

At $y = L$: $x(L) = \frac{X_0}{2} = \frac{F_x L^3}{48EI_z}$ ← B.C.

Substitute into (1):

$$\hat{x}(y) = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity:

$$v_b(y)\Big|_{[AB]} = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_0$$

Plugging into the expression for KE_b:

$$KE_{[AB]} = \frac{1}{2}\int_0^L \frac{X_0^2\omega_0^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

$$= \frac{X_0^2\omega_0^2 M_{[AB]}}{8L}\int_0^L \left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

$M_{[AB]}$ is the static beam mass

mass per unit length

$$K.E._{[AB]} = \frac{13}{280}X_0^2\omega_0^2 M_{[AB]}$$

mass modification factor

For segment [CD]:

$$v_y(y)\big|_{[CD]} = X_o\left[1-\frac{3}{2}\left(\frac{y}{L}\right)^2+\left(\frac{y}{L}\right)^3\right]\omega_o$$

Thus:

$$KE_{[CD]} = \frac{X_o^2\omega_o^2 M_{[CD]}}{2L}\int_0^L\left[1-\frac{3}{2}\left(\frac{y}{L}\right)^2+\left(\frac{y}{L}\right)^3\right]dy$$

$$\rightarrow KE_{[CD]} = \frac{83}{280}X_o^2\omega_o^2 M_{[CD]}$$

static mass of beam [CD]

Let $M_b \triangleq$ total mass of all 8 beams

Then: $M_{[AB]} = M_{[CD]} = \frac{1}{8}M_b$

Thus:

$$KE_b = 4KE_{[AB]} + 4KE_{[CD]} = \frac{6}{35}X_o^2\omega_o^2 M_b$$

and

$$KE_{max} = X_o^2\omega_o^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right]$$

$PE_{max} \rightarrow$ simply equal to work done to achieve maximum deflection (X_o)

$$PE_{max} = \frac{1}{2}k_x X_o^2$$

$k_c \leftarrow$ stiffness of a cantilever of length $\frac{1}{2}L$

Thus, using Rayleigh-Ritz:

$$KE_{max} = PE_{max}$$

$$\cancel{X_o^2}\omega_o^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right] = \frac{1}{2}k_x\cancel{X_o^2}$$

k_c

$$\boxed{\omega_o = \left[\frac{k_x}{M_{eq}}\right]^{1/2}}$$

$= k_c$

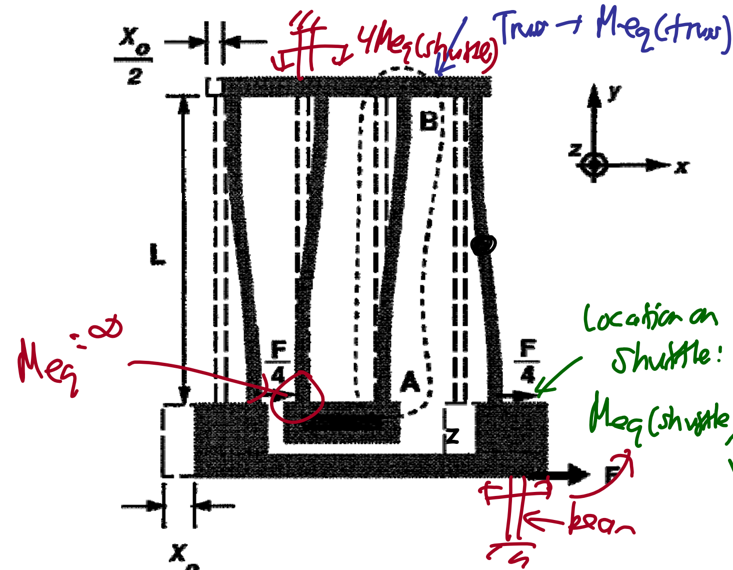
where $M_{eq} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b$

(Resonance Freq. of a Folded-Beam Suspended Shuttle)

- Looked briefly at Module 10 slides 21-31, but very quickly – you should go through it again on your own

Equivalent Dynamic Mass

Location on Folding Truss → $M_{eq(truss)}$ — *



Equivalent Mass:

for beam

$$\text{Equiv. Mass: } M_{eq,x} = \frac{KE_{max}}{\frac{1}{2}V_x^2} = \frac{\frac{1}{2}\rho A\int_0^{\ell} V^2(x)\,dx}{\frac{1}{2}V_x^2}$$

velocity @ location x

$$M_{eq(shuttle)} = \frac{KE_{max}}{\frac{1}{2}V_{shuttle}^2} = \frac{\cancel{\omega_o^2 x_o^2 \left(\frac{1}{2}\right)}\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]}{\cancel{\frac{1}{2}\omega_o^2 x_o^2}}$$

$$M_{eq(shuttle)} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b$$

static masses: ρ(volume)

$* \rightarrow$

$$M_{eq(truss)} = \frac{\cancel{\omega_o^2 x_o^2 \left(\frac{1}{2}\right)}\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]}{\cancel{\frac{1}{2}}\left(\frac{1}{4}\right)\cancel{\omega_o^2 x_o^2}}$$

$$M_{eq(truss)} = 4\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]$$

equiv. dynamic mass @ the truss location

Equiv. Dynamic Stiffness

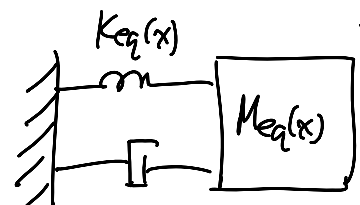
$$\omega_o = \sqrt{\frac{k_{eq}(x)}{M_{eq}(x)}} \rightarrow \boxed{k_{eq}(x) = \omega_o^2 M_{eq}(x)}$$

⇒ large equiv. mass & large stiffness go hand-in-hand

Equiv. Dynamic Damping

$$Q = \frac{\omega_o M_{eq}(x) \sim L}{c_{eq}(x) \sim R} \rightarrow c_{eq}(x) = \frac{\omega_o M_{eq}(x)}{Q} = \frac{\sqrt{k_{eq}(x) M_{eq}(x)}}{Q}$$

damping



specified @ single location, x

$c_{eq}(x) \rightarrow$ a $b_{eq}(x)$