Lecture 22: Pull-In Voltage

- Announcements:
- 2^{nd} project slide due this Friday (email it)
  - Hypothesis
  - Identify pros and cons of scaling
- Module 12 online
- HW#5 due tomorrow morning, BUT 8 PM ON FRIDAY FOR THE LAYOUT
- HW#6 online soon
- This lecture will be 2 hours to make up for the technical difficulties last time
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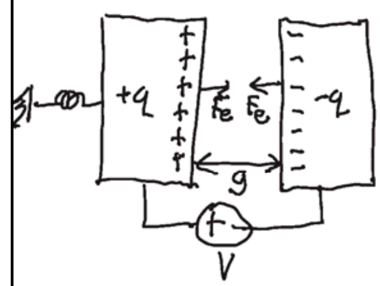
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1^{st} Order Analysis
    - 2^{nd} Order Analysis
- -------------------------------------
- Last Time:
- Started energy conserving transducers

over

Energy Conserving Transducers



Goal: Determine gap spacing g as a function of input variables $V, I \Rightarrow q$

Assume the plates are supported elastically

How do we proceed?
↳ ① Determine the energy of the system.
② What can I Δ to Δ the energy in the system?

(i) change the charge q

(ii) change the separation g

$$\Delta W(q,g) = V\Delta q + F_e \Delta g$$

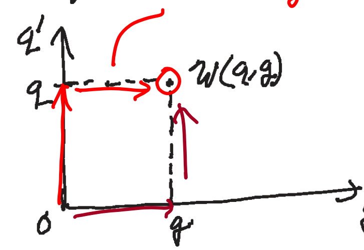
$$dW = Vdq + F_e dg$$

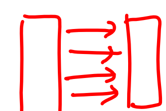
hold q = const. → $Vdq \to 0$

$dW = F_e dg \rightarrow F_e = \left.\dfrac{dW}{dg}\right|_{q=\text{const.}}$

Stored Energy

No change in charge q: $dq = 0$





$+q \quad \left(-q \quad \mathcal{E} = \dfrac{q}{\varepsilon A} \right)$

$$W = 0 + \int_0^g F_e \, dg'$$

$$F_e = \left(\frac{q}{2}\right)\mathcal{E} = \frac{1}{2}\frac{q^2}{\epsilon A} \quad \text{(independent of } g\text{)}$$

$$\therefore W = \int_0^g F_e \, dg' = F_e g' \Big|_0^g = F_e g$$

$$\boxed{W(g) = \frac{1}{2}\frac{q^2}{\epsilon A} g}$$

⇒ Now, look at work done to charge a C to q at fixed gap g. $g \to 0$

$$dW = V dq + F_e \cancel{dg}$$

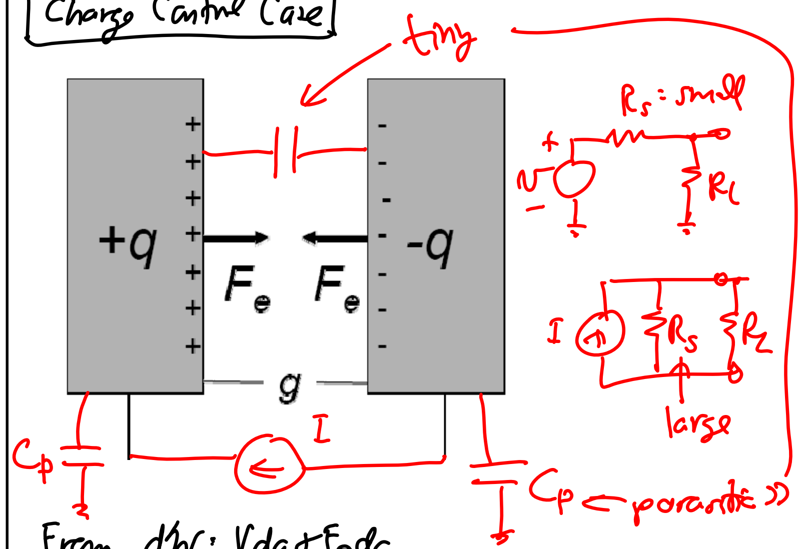
For a capacitor: $q = CV \rightarrow V = \frac{q}{C}$

$$\therefore W(q) = \int_0^q V dq = \int_0^q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{q^2}{C}$$

$$= \boxed{\frac{1}{2}\frac{q^2}{\epsilon A} g = W(q)}$$

$=$

Charge Control Case



From $dW = V dq + F_e dg$

⇒ Force is given by

$$F_e = \frac{\partial W(q,g)}{\partial g}\Big|_{q=const.} = \frac{\partial}{\partial g}\left(\frac{1}{2}\frac{q^2}{\epsilon A} g\right)$$

$$\therefore \boxed{F_e = \frac{1}{2}\frac{q^2}{\epsilon A}} \Rightarrow \text{indep. of gap spacing!}$$

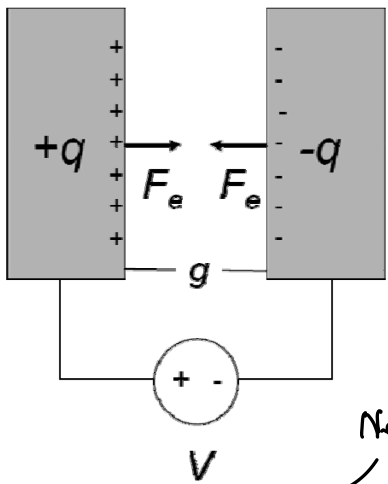
⇒ voltage is given by

$$V = \frac{\partial W(q,g)}{\partial q}\Big|_{g=const.} = \frac{\partial}{\partial q}\left(\frac{1}{2}\frac{q^2}{\epsilon A} g\right)$$

$$V = \frac{qg}{\epsilon A} \rightarrow \boxed{V = \frac{q}{C}}$$

(consistent w/ what we already know)

Voltage Control



Want to write $F_e = f(V)$

We know this:

$dW = Vdq + F_e dg$

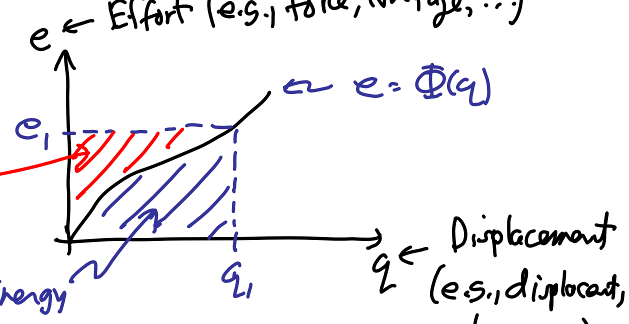
$W = f(q, g)$

Need $W'(V, g)$

want to replace q w/ V

Can get this using a Legendre Transformation.

Energy & Co-Energy



$W(q_1) = \int_0^{q_1} e\,dq = \int_0^{q_1} \Phi(q)\,dq \quad (1)$

Co-Energy:

$W'(e_1) = \int_0^{e_1} q\,de = \int_0^{e_1} \Phi^{-1}(e)\,de \quad (2)$

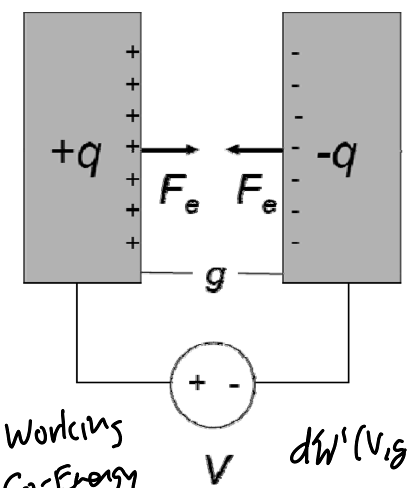
For a linear system, (1) & (2) will be equal.

Can define co-energy as:

$W'(e) = eq - W(q)$ (from the plot)

co-energy

Co-Energy Formulation for Voltage Control



$W'(V, g) = Vq - W(q, g)$

Differentially, this becomes:

$dW'(V, g) = (q\,dV + \cancel{V\,dq}) - dW(q, g)$

$[dW(q, g) = F_e\,dg + \cancel{V\,dq}]$

Working Co-Energy Expression

$\boxed{dW'(V, g) = q\,dV - F_e\,dg}$

Find co-energy in terms of voltage, V:

$$W' = \int_0^V q(g,V')\,dV' = \int_0^V \left(\frac{\epsilon A}{g}\right) V'\,dV'$$

$$= \frac{1}{2}\left(\frac{\epsilon A}{g}\right)V^2 = \frac{1}{2}CV^2 \checkmark \quad \text{(as expected)}$$

Electrostatic (a Voltage-Controlled) Force:

$$F_e = -\left.\frac{\partial W'(V,g)}{\partial g}\right|_{V=\text{const}}$$

$$= -\left(-\frac{1}{2}\right)\left(\frac{\epsilon A}{g^2}\right)V^2 = \boxed{\frac{1}{2}\frac{C}{g}V^2 = F_e}$$

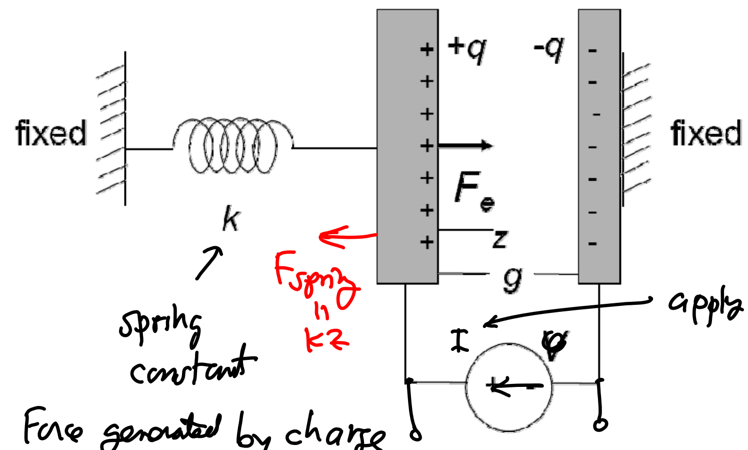
depends on gap!

Charge:

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_{g=\text{const.}} = \frac{\epsilon A}{g}V = CV \checkmark$$

(as expected)

Charge Control of a Spring-Suspended C



Force generated by charge q (supplied by current I).

$$F_e = \left.\frac{\partial W(q,g)}{\partial g}\right|_q = \frac{q^2}{2\epsilon A}$$

Restoring force of the spring: $F_{spring} = kz = F_e$ ← equilibrium

The gap:

$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k} = g}$$

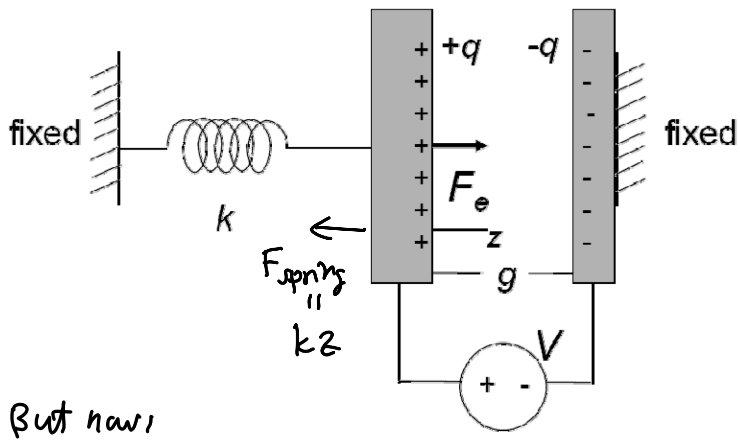
g_o: initial gap

$q\uparrow$ can drive $g \to 0$ in a continuous fashion

$$V = \frac{q}{C} = \frac{q}{\epsilon A}g = \boxed{\frac{q}{\epsilon A}\left(g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k}\right) = V}$$

→ $V\downarrow$ as $q\downarrow$

Voltage-Control of a Suspended C



But now:

$$F_e = \left.\frac{\partial W'(V,g)}{\partial g}\right|_q \Rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{\epsilon A}{g^2}\frac{V^2}{k} = g}$$

initial gap spacing

g shows up on both sides!

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$

(+) Feedback!

If loop gain > 1, then this will go unstable!

plate will collapse into the electrode!

(BOOM!)

Charge: (for a stable gap)

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_g = CV \checkmark \quad \text{(as expected)}$$

Stability Analysis

⇒ determine under what conditions voltage-control will cause a collapse of the plates

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_0 - g)}_{F_{spring}}$$

What happens when I change g by a small increment dg?

get an increment in the net attractive force, F_{net}

$$dF_{net} = \frac{\partial F_{net}}{\partial g}dg = \left[-\frac{\epsilon A V^2}{g^3} + k\right]dg$$

(−) (−)

If $g\downarrow \rightarrow dg = (-)$, then for stability need $F_{net}\downarrow \rightarrow dF_{net} = (-)$

This needs to be (+)! → otherwise, the plates collapse!

Thus: $\boxed{k > \frac{\epsilon A V^2}{g^3}}$ (for a stable uncollapsed system)

Pull-in Voltage & Pull-in Gap

$V_{PI} \triangleq$ voltage @ which the plates collapse

$g_{PI} \triangleq$ gap @ " " " "

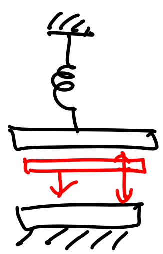
The plates go unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \quad (1)$$

$$F_{net} = 0 = \frac{\epsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_0 - g_{PI}) \quad (2)$$

Substitute (1) into (2):

$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2 \cancel{g_{PI}^2}} - \frac{\cancel{\epsilon A V_{PI}^2}}{g_{PI}^{\cancel{3}}}(g_0 - g_{PI})$$

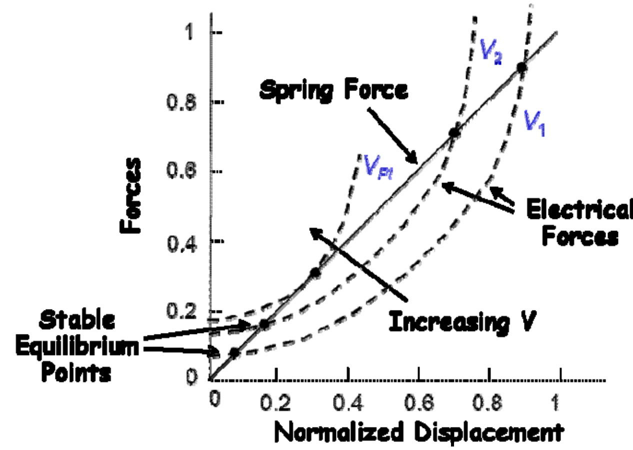


$$\frac{g_0 - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_0 = \frac{3}{2} g_{PI}$$

$$\therefore \boxed{g_{PI} = \frac{2}{3} g_0}$$

when the gap is driven by a voltage to (2/3) the initial gap → collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \rightarrow \boxed{V_{PI} = \sqrt{\frac{8}{27}\frac{k g_0^3}{\epsilon A}}} \leftarrow \text{pull-in voltage}$$



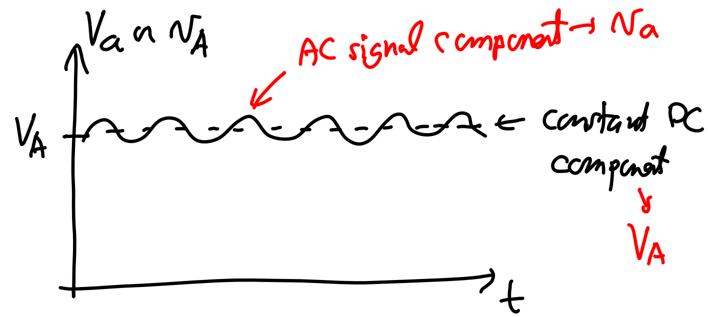
Advantages of Electrostatic Actuators:

- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

Disadvantages of Electrostatic Actuators:

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale

$\boxed{\text{Variable Nomenclature}}$



$V_a = v_A = V_A + v_a$

$v_\$ \quad v_¢$