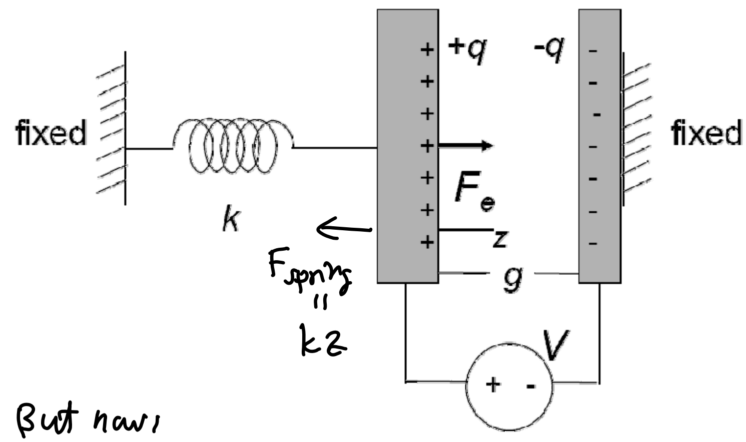
Lecture 22: Pull-In Voltage

- Announcements:
- 2nd project slide due this Friday (email it)
  - Hypothesis
  - Identify pros and cons of scaling
- Module 12 online
- HW#5 layout due at 8 PM ON FRIDAY
- HW#6 online
- This lecture will be 2 hours to make up for the technical difficulties of a previous lecture
- -------------------------------------
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
- -------------------------------------
- Last Time:
- Derived the pull-in voltage

over

Voltage-Control of a Suspended C



But now:

$$F_e = \frac{\partial W'(V,g)}{\partial g}\Big|_q \Rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{\epsilon A}{g^2}\frac{V^2}{k} = g}$$

initial gap spacing

g shows up on both sides!

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$

(+) Feedback!

If loop gain > 1, then this will go unstable!

plate will collapse into the electrode!

(BOOM!)

Charge: (for a stable gap)

$$q = \frac{\partial W'(V,g)}{\partial V}\Big|_g = CV \checkmark \quad \text{(as expected)}$$

Stability Analysis

⇒ determine under what conditions voltage-control will cause a collapse of the plates

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_0 - g)}_{F_{spring}}$$

What happens when I change g by a small increment dg?

↳ get an increment in the net attractive force, F_{net}

$$\overset{(-)}{dF_{net}} = \frac{\partial F_{net}}{\partial g} dg = \left[-\frac{\epsilon A V^2}{g^3} + k\right] \underset{(-)}{dg}$$

If $g\downarrow \rightarrow dg = (-)$, then for stability need $F_{net}\downarrow \rightarrow dF_{net} = (-)$

This needs to be (+)! → otherwise, the plates collapse!

Thus: $\boxed{k > \frac{\epsilon A V^2}{g^3}}$ (for a stable uncollapsed system)

Pull-in Voltage & Pull-in Gap

$V_{PI} \triangleq$ voltage @ which the plates collapse

$g_{PI} \triangleq$ gap @ " " " "

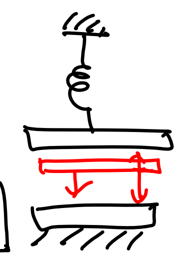
The plates go unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \quad (1)$$

$$F_{net} = 0 = \frac{\epsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_0 - g_{PI}) \quad (2)$$

Substitute (1) into (2):

$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2 \cancel{g_{PI}^2}} - \frac{\cancel{\epsilon A V_{PI}^2}}{g_{PI}^{\cancel{3}}}(g_0 - g_{PI})$$

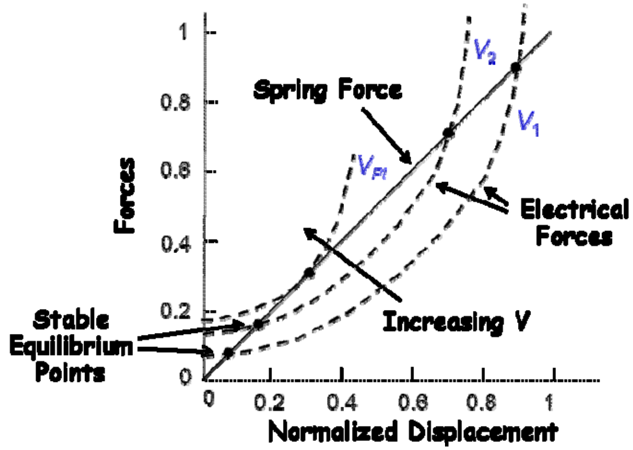


$$\frac{g_0 - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_0 = \frac{3}{2} g_{PI}$$

$$\therefore \boxed{g_{PI} = \frac{2}{3} g_0}$$

when the gap is driven by a voltage to (2/3) the initial gap → collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \rightarrow \boxed{V_{PI} = \sqrt{\frac{8}{27}\frac{k g_0^3}{\epsilon A}}}$$ ← pull-in voltage



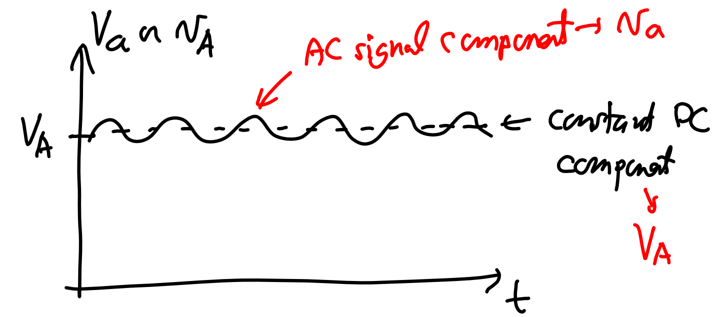
Advantages of Electrostatic Actuators:

- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

Disadvantages of Electrostatic Actuators:

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale

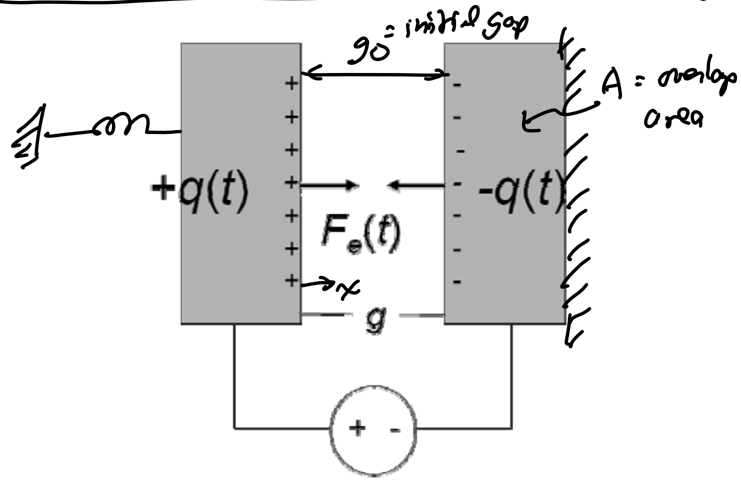
$\boxed{\text{Variable Nomenclature}}$



$V_a = v_A = V_A + v_a$

$v_{\$}$ $v_{¢}$

Linearizing the Voltage-to-Force Transfer Function



$$v(t) = V_P + v_i(t)$$

DC small-signal AC

$$F_e(t) = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}C[v(t)]^2\right]$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}\left[V_P + v_i(t)\right]^2$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}\left[V_P^2 + 2V_P v_i(t) + \cancel{[v_i(t)]^2}\right]$$

negl.

$$[V_P \gg v_i(t)] \Rightarrow F_e(t) = \underbrace{\frac{1}{2}V_P^2\frac{\partial C}{\partial x}}_{\text{DC Offset}} + \underbrace{V_P\frac{\partial C}{\partial x}v_i(t)}_{\text{AC Drive Signal}}$$

$$C_0 = \frac{\epsilon A}{g_0}$$

$$C(x) = \frac{\epsilon A}{g_0 - x} = C_0\left(1 - \frac{x}{g_0}\right)^{-1}$$

$$[x \ll g_0] \Rightarrow \approx C_0\left(1 + \frac{x}{g_0}\right)$$

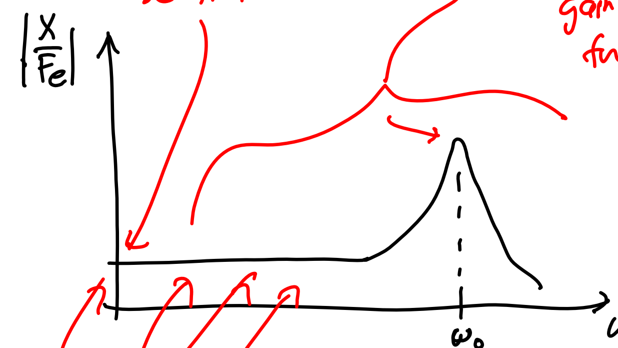
$$\therefore \boxed{\frac{\partial C}{\partial x} = \frac{C_0}{g_0} = \frac{\epsilon A}{g_0^2}}$$

$$\Rightarrow F_e(t) = \underbrace{\frac{1}{2}\frac{C_0}{g_0}V_P^2}_{\text{DC offset}} + \underbrace{V_P\frac{C_0}{g_0}v_i(t)}_{\text{AC term}}$$

gain control for ac signals

gain ~ constant for small signals

linear dependence



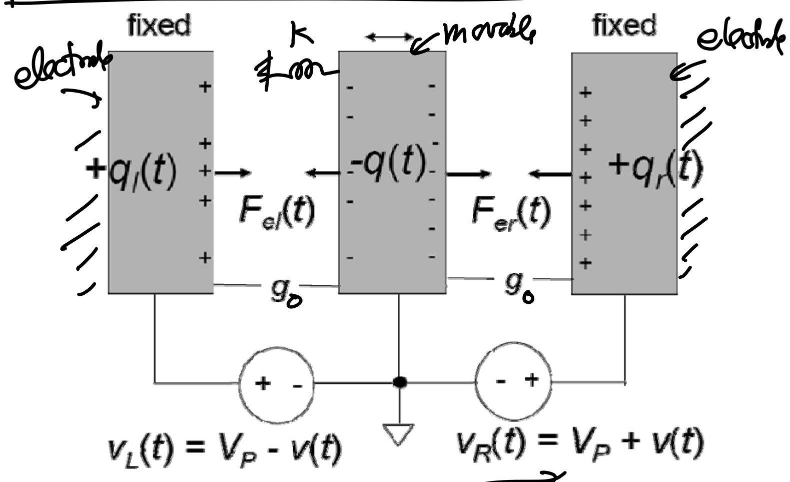
very small response

But: must still worry about $\underbrace{\text{pull-in}}_{}$

Determines the V_P limit

Can we fix this?

Cancel the DC Offset Using Differential Symmetry



differential input into balanced structure

$$F_{net} = F_{er}(t) - F_{el}(t)$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{[v_R(t)]^2 - [v_L(t)]^2\right\}$$

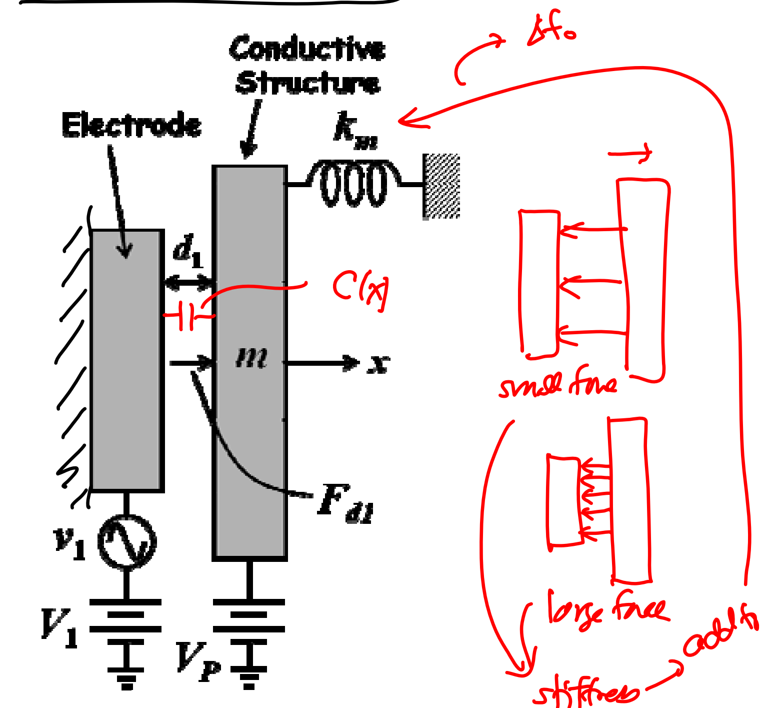
$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{\cancel{V_P^2} + 2V_P v(t) + \cancel{[v(t)]^2} - \left(\cancel{V_P^2} - 2V_P v(t) + \cancel{[v(t)]^2}\right)\right\}$$

$$\therefore \boxed{F_{net}(t) = 2V_P\frac{\partial C}{\partial x}v(t) = 2V_P\frac{C_o}{g_o}v(t)}$$

↳ No V_P component → no pull-in (ideally)

Reality: $V_P\uparrow$, but not infinite

Nonlinearity Still Effects Us



More Complete Expression

$$C_1(x) = \frac{\epsilon A}{d_1 + x} = C_{o1}\left(1 + \frac{x}{d_1}\right)^{-1} \rightarrow \frac{\partial C_1}{\partial x} = -\frac{C_{o1}}{d_1}\left(1 + \frac{x}{d_1}\right)^{-2}$$

[Expand into Taylor Series]

$$\frac{\partial C_1}{\partial x} = -\frac{C_{o1}}{d_1}\left(1 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots\right)$$

where $A_1 = -\frac{2}{d_1}$, $A_2 = \frac{3}{d_1^2}$, $A_3 = -\frac{4}{d_1^3}$, ...

$$F_{d1} = \frac{1}{2}\frac{\partial C_1}{\partial x}(V_P - V_1 - v_1)^2 = \frac{1}{2}\frac{\partial C_1}{\partial x}(V_{P1} - v_1)^2$$

$V_{P1} = V_P - V_1$

[small displacements: $x \ll d_1$]

$\cos^2\omega t = \frac{1}{2}(1+\cos 2\omega t)$

$$F_{d1} = \frac{1}{2}\left(-\frac{C_{01}}{d_1}\right)(1+A_1 x)(V_{P1}^2 - 2V_{P1}v_1 + v_1^2)$$

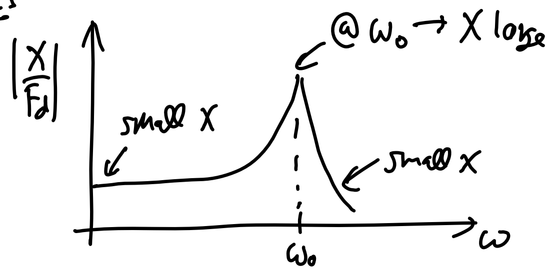
@ freq. of v_1

$$= \frac{1}{2}\left(-\frac{C_{01}}{d_1}\right)\left\{V_{P1}^2 - 2V_{P1}v_1 + v_1^2 + A_1 V_{P1}^2 x - 2A_1 V_{P1} x v_1 + A_1 x v_1^2\right\}$$

← also signal component

$\cancel{\omega\ \omega}$

Resonance



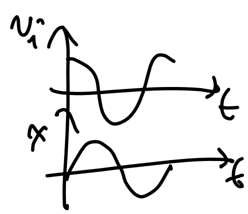
@ resonance

$$x = \frac{QF_{d1}}{jk} = \frac{Q}{jk}\frac{\partial C_1}{\partial x}V_{P1}v_1$$

90° phase shift

$$v_1 = |v_1|\cos\omega_0 t \longrightarrow x = |x|\sin\omega_0 t$$

90° phase shift



Force term @ ω_0

$$F_{d1}\big|_{\omega_0} = \underbrace{V_{P1}\frac{C_{01}}{d_1}|v_1|\cos\omega_0 t}_{\text{Drive Force Term}} + \underbrace{V_{P1}^2\frac{C_{01}}{d_1^2}}_{k_e \rightarrow \text{electrical stiffness}}|x|\sin\omega_0 t$$

proportional to x

90° phase-shifted f/

∴ in phase w/ displacement!

∴ it's a stiffness!

Electrical Stiffness:

① A negative spring constant!

② Derives from V_P:

$$\boxed{k_e = V_{P1}^2\frac{C_{01}}{d_1^2} = V_{P1}^2\frac{\epsilon A}{d_1^3}}$$

overlap area of C

DC Bias

3rd power dependence on gap!

k_e → can affect resonance freq, f_0!

$\omega_0 \triangleq$ radian resonance freq. w/ no V_P applied ($V_{P1} = 0V$)

$$\omega_0' = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_m - k_e}{m}} \quad *$$

$$\omega_0 = \sqrt{\frac{k_m}{m}}$$

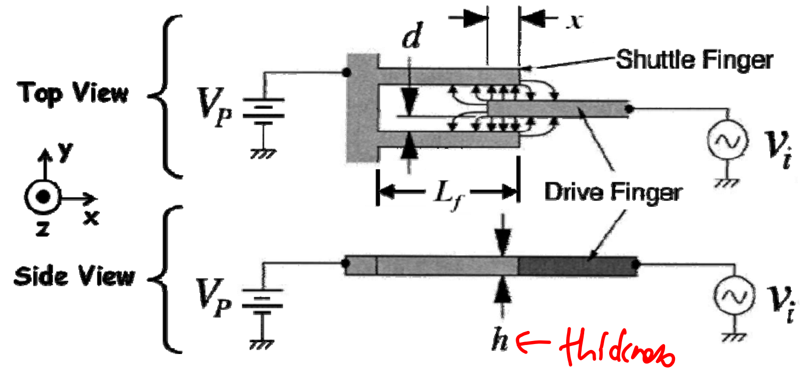
$$* = \sqrt{\frac{k_m}{m}}\left(1-\frac{k_e}{k_m}\right)^{1/2}$$

$$\boxed{\omega_0' = \omega_0\left[1-\frac{V_{P1}^2}{k_m}\frac{\epsilon A}{d_1^3}\right]^{1/2}}$$

Now a fcn of DC bias voltage!
(voltage-controllable!)

- Go through Module 12 slides 26-35

Electrostatic Comb-Drive



$$F_d = \frac{\partial W'}{\partial x} = \frac{1}{2}\frac{\partial C}{\partial x}(V_P - v_i)^2$$

Need $C(x)$

$$C(x) = \frac{2\epsilon_o h x}{d} \rightarrow \frac{\partial C}{\partial x} = \frac{2\epsilon_o h}{d} \leftarrow \text{not a fcn of } x!$$

↓ ∴ no electrical stiffness!

$$F_d = \frac{1}{2}\frac{2\epsilon_o h}{d}\left[\cancel{V_P^2} - 2V_P v_i + \cancel{v_i^2}\right]$$

can balanced out by symmetrically placed electrodes

$v_i \ll V_P$

$$\boxed{F_d = -2V_P\frac{\epsilon_o h}{d}v_i}$$

- Go through the rest of Module 12, starting from slide 38
- Start Module 13 and go through slides 1-9