PROBLEM SET 1

Issued: Monday, Feb. 2, 2015

Due (at 9 a.m.): Friday, Feb. 13, 2015, in the EE C247B HW box near 125 Cory.

This homework assignment is intended to give you some early practice playing with dimensions and exploring how scaling can greatly improve or degrade certain performance characteristics of mechanical systems. Don't worry at this point if you do not understand fully some of the physical expressions used. They will be revisited later in the semester. This assignment just gives you a chance to play with them a bit.

1. Photons, just like any other particle, carry momentum. This means a beam of light can actually exert an impulsive force on an object upon reflection. This problem examines a cantilever system to detect radiation pressure force. In particular, consider the macro scale fixed-free beam (cantilever) show in Fig. PS1.1, constructed of glass (Young's modulus E = 75 GPa, density ρ = 2200 kg/m^3) with dimensions L = 1 m, W = 10 cm, H = 5 cm. An input light beam from a laser (with wavelength λ = 1550 nm) hits the tip of this cantilever-shaped mirror and reflects back from the surface.

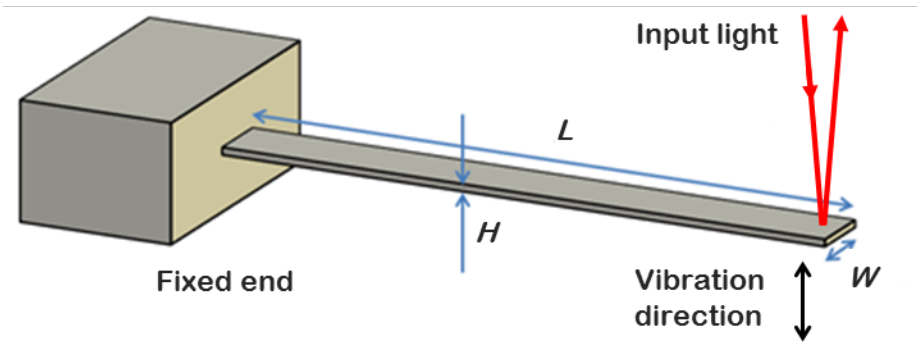

Figure PS1.1

(a) Calculate the mechanical resonance frequency, *f*, and the stiffness, *k*, of the cantilever using the following formulas:

$$f = \frac{\frac{1}{2\pi}\sqrt{\frac{35}{33}}\sqrt{\frac{E}{\rho}}\,H}{L^2}, \quad k = \frac{1}{4}EW\frac{H^3}{L^3} \tag{1}$$

where E and ρ are the Young's modulus and density of the structural material, and dimensions are given in Figure PS1.1.

(b) Thermal noise sets a limit on the minimum detectable force using this cantilever. Find the root-mean-square value of the minimum detectable force using:

$$F_{min} = \sqrt{\frac{2k\Delta f k_B T}{\pi Q f}} \tag{2}$$

where k_B is the Boltzmann constant, T is temperature, Q is the quality factor associated with the mechanical resonance, and Δf is the measurement bandwidth. Assume Q = 1,000 and the measurement bandwidth is 5 Hz.

(c) The radiation pressure exerted on the mirror by a reflecting photon can be determined from:

$$F_{rp} = \frac{dp}{dt} \tag{3}$$

where p is momentum. Each photon carries a momentum $p = \frac{h}{\lambda}$ (DeBroglie theorem) and reflects from the mirror with the same momentum, but in the opposite direction. Derive the expression for the radiation pressure force experienced by the mirror assuming input coherent light with a photon arrival rate (number of photons per second) of s_{in}. (Hint: This would be the product of Δp and s_{in}) .

(d) Find the minimum detectable laser power using the minimum detectable force calculated in part (b).

(e) Now assume that the mirror dimensions are scaled to L = 50 µm, W = 5 µm, H = 2 µm. Recalculate (b) and (d) assuming Q and Δf don't change.

(f) Suppose the light is removed, and the cantilever is repurposed to detect a harmful chemical agent by applying a special polymer on its tip to which a monolayer of the chemical agent can bond. When this bonding occurs, the tip's mass increases slightly, inducing a corresponding negative shift in the beam's resonance frequency. Recall that $\omega = \sqrt{k/m}$ for a simple mass/spring oscillator. Assuming the shift in frequency is caused only by the change in the mass at the tip, will the fractional frequency shift ($\Delta f/f$) be greater for a thicker cantilever (larger h) or a thinner cantilever (smaller h)? Why?

(g) Assume that the tip of the cantilever is excited at its resonant frequency and that its motion is sensed by an electrode placed over the last 1/8 of the cantilever length near its tip, as shown in Figure PS1.2.

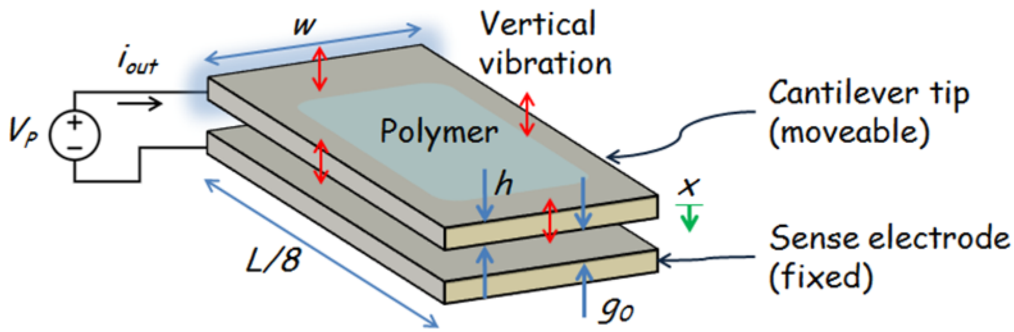

Figure PS1.2

If one assumes vertical cantilever displacements small enough to neglect rotation of the cantilever as it moves up and down, the sense electrode and cantilever tip form an approximate parallel plate capacitor with a gap that depends on the displacement of the tip. The capacitance as a function of displacement is then given by $C(x) = \frac{\epsilon_0 wL/8}{g_0 - x}$, where $\epsilon_0 = 8.854 \cdot 10^{-12}$ F/m is the electrical permittivity of free space, $wL/8$ is the overlap area of the plates forming the capacitor, g_0 is the original gap spacing with no displacement, and x is the displacement of the cantilever tip. To sense the time varying capacitance, a "polarizing voltage" V_p is applied between the cantilever and sensing plate. This places electric charge on the capacitor given by $q = C(x)V_p$. Since $i_{out} = \frac{dq}{dt} = C(x)\frac{dV_p}{dt} + V_p\frac{dC(x)}{dt}$ and V_p is constant, $i_{out} = V_p \frac{dC(x)}{dx}\frac{dx}{dt}$. From this equation, the output current magnitude is proportional to three variables: V_p, $\frac{dC(x)}{dx}$ and $\frac{dx}{dt}$.

i. Write the output current expression in phasor form.

ii. Find an expression for $\frac{dC(x)}{dx}$. Then simplify this expression assuming $x \ll g_0$.

iii. Assuming a constant displacement amplitude, suggest five variables that can be scaled to obtain a larger output current. Also indicate if they should be increased or decreased. Which one of them would you scale if you were allowed to choose only one?

(h) Assume that a 150 mm (6″) diameter wafer has a useful area of 100 mm × 100 mm over which cantilever sensors can be fabricated. (Here, the edges of the wafer are for handling, so do not yield working devices.) A dicing saw is used to cut the wafer into individual dies and the width of each cut is 50 µm. Each sensor requires a square unit cell with a minimum area of $9L^2$. The cost per sensor is given by $C(n,d) = (\$3000 + \$1 \times n + \$2 \times d)/d$, where *n* is the number of cuts through the wafer and *d* is the number of dies. Here, the fixed $2 cost per sensor derives from post processing, packaging and testing costs. Assume that the minimum die size that can be reliably handled is 1 mm × 1 mm. What is the lowest achievable fabrication cost per sensor (to the nearest cent) and what is the corresponding maximum cantilever size (to the nearest ten microns)? [Hint: It would be helpful to define *d(n)*, then find *n*.]

2. Suppose you are asked to design a polycrystalline silicon clamped-clamped beam resonator, such as discussed in lecture. For polycrystalline silicon, assume the following material properties: Young's modulus E = 150 GPa, density ρ = 2300 kg/m^3, and Poisson ratio ν = 0.226.

(a) Consider a beam with width W_r = 8 μm and thickness h = 2 μm. Use Euler-Bernoulli theory (i.e., the formulation covered in class) to determine the length of the beam L_r that allows it to mechanically resonate in a direction perpendicular to the substrate at:

(i) 10 MHz, (ii) 100 MHz, (iii) 1 GHz.

(b) Use Euler-Bernoulli theory to determine the length of the beam L_r that allows it to mechanically resonate perpendicular to the substrate at 1 GHz if the beam width W_r and thickness h are as follows: (i) W_r = 8 μm, h = 2 μm; (ii) W_r = 1 μm, h = 1 μm; and (iii) W_r = 300 nm, h = 100 nm.

(c) Euler-Bernoulli theory is actually not very accurate when the length of the beam begins to approach its thickness, mainly because it ignores shear displacements and rotary inertias. (These are things that you will learn more about later in the course.) For cases where thickness approaches length, the more complicated Timoshenko design procedure should be used to model a beam's resonance characteristics. For a clamped-clamped beam, Timoshenko's design procedure uses the following equation:

$$\tan\frac{\beta}{2} + \frac{\beta}{\alpha}\left(\frac{\alpha^2 + g^2\left(\frac{\kappa G}{E}\right)}{\beta^2 - g^2\left(\frac{\kappa G}{E}\right)}\right)\tanh\frac{\alpha}{2} = 0 \qquad (4)$$

where

$$g^2 = \omega_o^2 L_r^2\left(\frac{\rho}{E}\right) \qquad (5)$$

$$\alpha^2 = \frac{g^2}{2}\left[-\left(1+\frac{E}{\kappa G}\right) + \sqrt{\left(1-\frac{E}{\kappa G}\right)^2 + \frac{4L_r^2 h W_r}{g^2 I_r}}\right]$$
$$\beta^2 = \frac{g^2}{2}\left[+\left(1+\frac{E}{\kappa G}\right) + \sqrt{\left(1-\frac{E}{\kappa G}\right)^2 + \frac{4L_r^2 h W_r}{g^2 I_r}}\right] \qquad (6)$$

$$I_r = \frac{W_r h^3}{12} \qquad (7)$$

$$G = \frac{E}{2(1+\nu)} \qquad (8)$$

and where for a rectangular beam, κ = 2/3.

Use Timoshenko's formulas above to determine the actual frequencies of the beams you designed in parts (a) and (b) above. Can you suggest a rule for scaling of beams to attain higher frequencies that insures Euler-Bernoulli theory works reasonably well?

3. Suppose a step function voltage V_A was suddenly applied across the anchors of a 2 μm thick polysilicon beam and proof mass setup as shown in Figures PS1.3-1 and PS1.3-2, which also provide lateral dimensions. For polysilicon, assume the following material properties: Young's modulus E = 150 GPa, density ρ = 2300 kg/m^3, Poisson ratio ν = 0.226, sheet resistance (resistivity•thickness^{-1}) = 10 Ω/□, specific heat = 0.77 J/(g•K), and thermal conductivity = 30 W/(m•K).

(a) With what time constant will the proof mass reach its steady-state temperature after the voltage V_A steps from 0V to 1V? Give a formula and a numerical answer with units.

(b) If the final step function value of V_A is 1V, what is the steady-state temperature of the proof mass? Give a formula and a numerical answer with units.

(c) What effect do you think the applied voltage has on the resonance frequency of the structure in the z-direction (into the page)? Give a brief qualitative explanation.

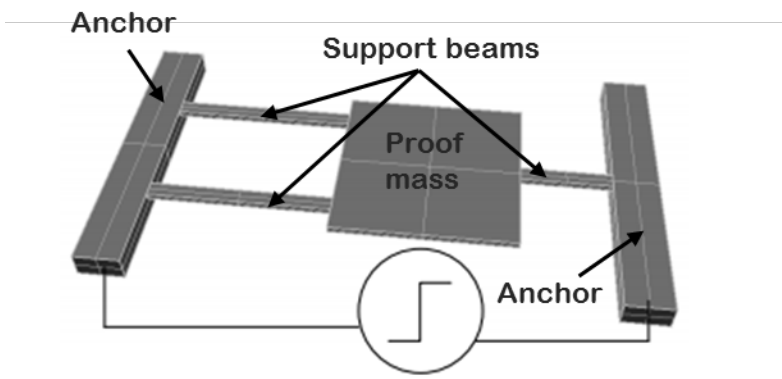

Figure PS1.3-1

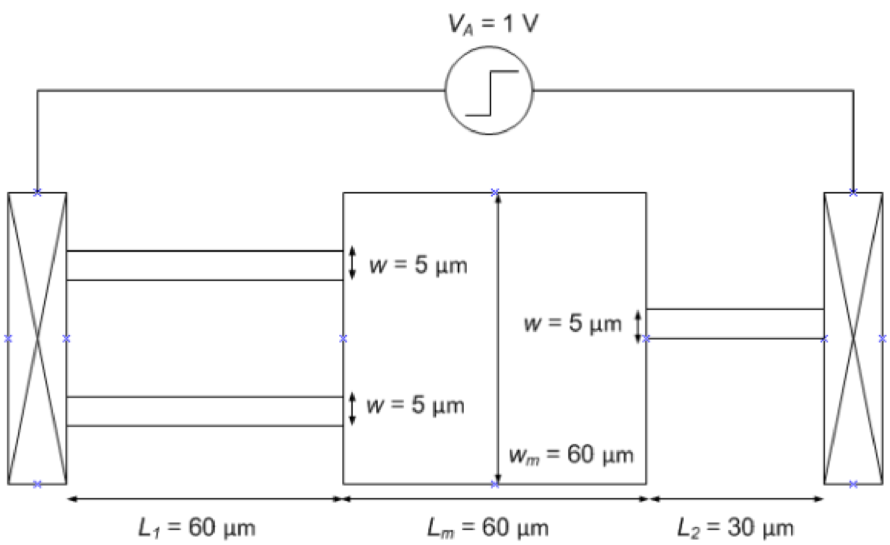

Figure PS1.3-2