Lecture 10: Beam Bending

- Announcements:
- HW#2 due tomorrow morning
- Lecture Module 8 on Microstructural Elements online
- Again, this will be a 2 hour lecture
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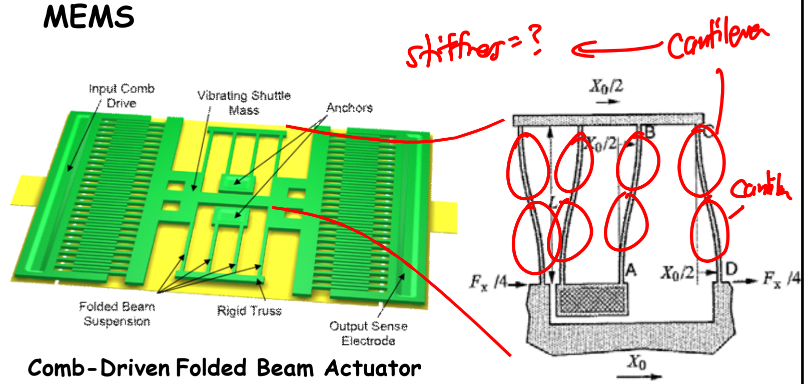
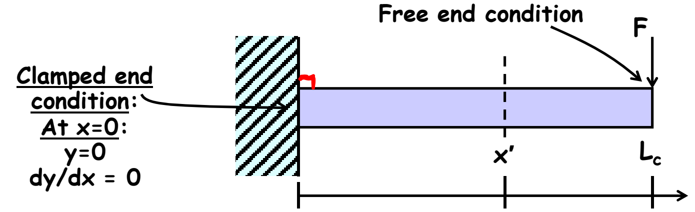
- Reading: Senturia, Chpt. 8
- Lecture Topics:
  - Stress, strain, etc., for isotropic materials
  - Thin films: thermal stress, residual stress, and stress gradients
  - Internal dissipation
  - MEMS material properties and performance metrics
- -----------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- -----------------------------------
- Last Time:
- Went through Module 7 on Mechanics of Materials
- Now finish this
- Then, start a new topic: Bending of Beams

- Springs and suspensions very common in MEMS
- Coils are popular in the macro-world; but not easy to make in the micro-world
- Beams: simpler to fabricate and analyze; become "stronger" on the micro-scale → use beams for MEMS

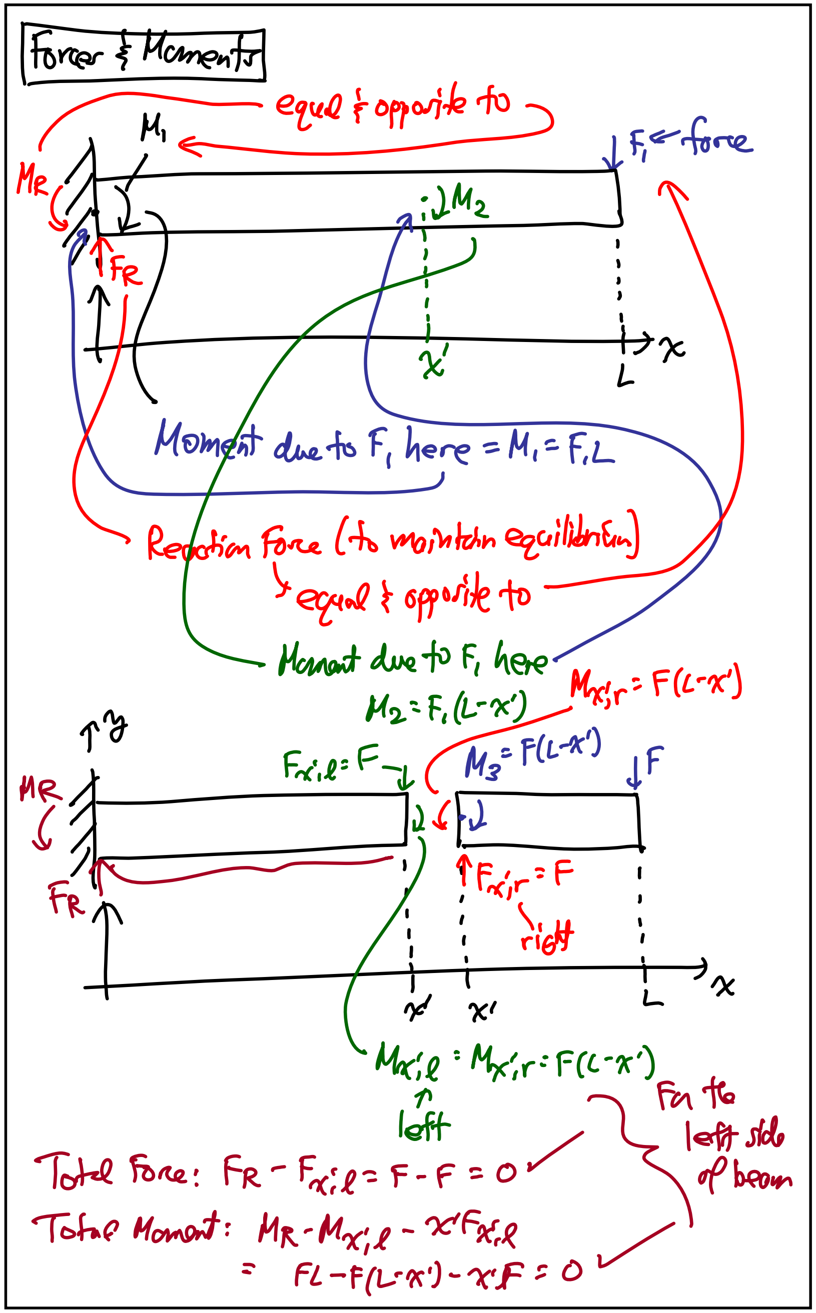


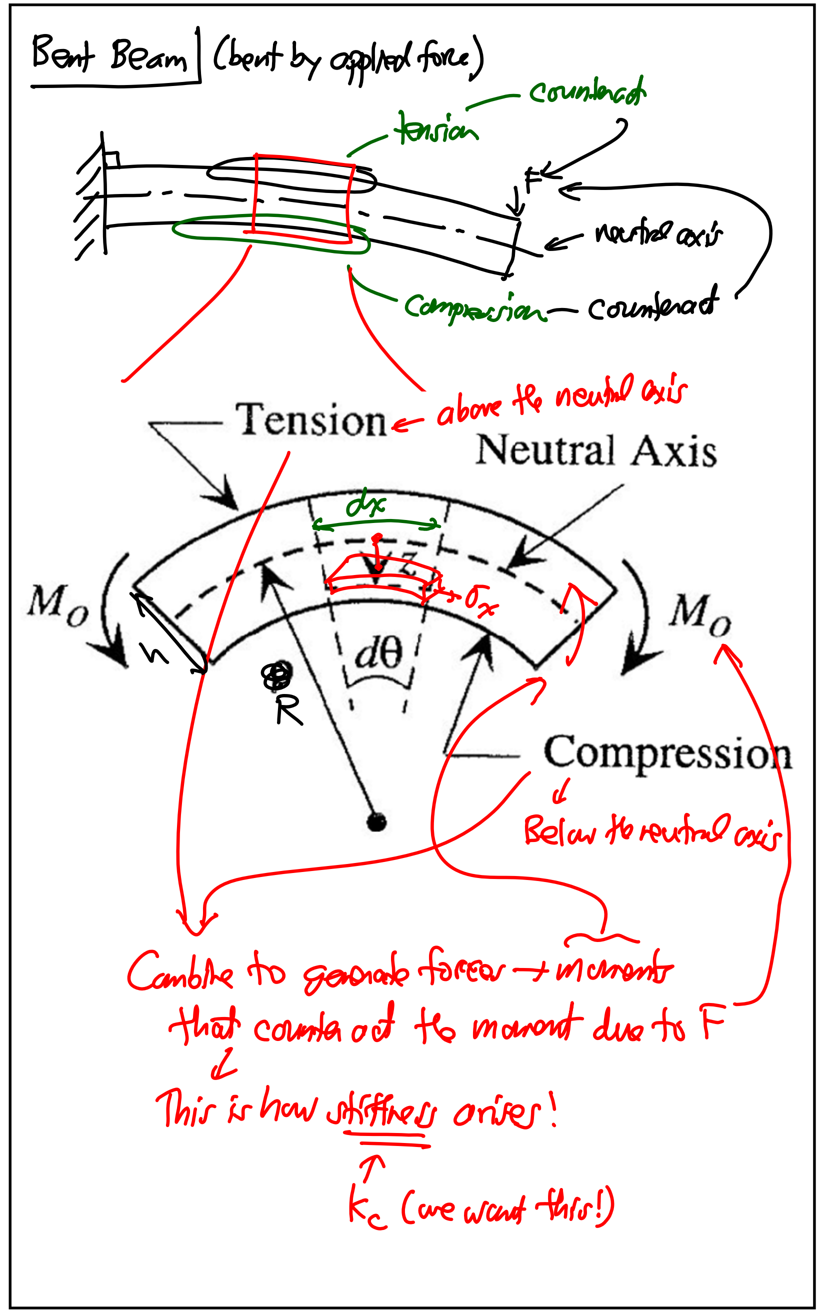
Comb-Driven Folded Beam Actuator

Problem: Bending a Cantilever Beam



- Objective: Find relation between tip deflection $y(x=L_c)$ and applied load F
  want $F = k_c x$, $k_c = ?$
- Assumptions:
  1. Tip deflection is small compared with beam length
  2. Plane sections (normal to beam's axis) remain plane and normal during bending, i.e., "pure bending"
  3. Shear stresses are negligible


Forces & Moments
equal & opposite to
M_1
F_1 ← force
M_R
M_2
F_R
x
x'
L
Moment due to F_1 here $= M_1 = F_1 L$
Reaction Force (to maintain equilibrium)
equal & opposite to
Moment due to F_1 here
$M_2 = F_1(L-x')$
$M_{x',r} = F(L-x')$
$F_{x',l} = F$
$M_3 = F(L-x')$
F
y
$F_{x',r} = F$
right
$M_{x',l} = M_{x',r} = F(L-x')$
left
For the left side of beam
Total Force: $F_R - F_{x',l} = F - F = 0$
Total Moment: $M_R - M_{x',l} - x' F_{x',l}$
$= FL - F(L-x') - x'F = 0$


Bent Beam (bent by applied force)
counteract
tension
F
neutral axis
compression — counteract
Tension ← above the neutral axis
Neutral Axis
dx
z
σ_x
M_o
M_o
h
$d\theta$
R
Compression
Below the neutral axis
Combine to generate forces → moments
that counteract the moment due to F
This is how stiffness arises!
k_c (we want this!)

Beam Segment in Pure Bending

⇒ Consider the segment bounded by the dashed lines defining $d\theta$

At $z=0$: neutral axis → segment length $= dx = R\,d\theta$ (1)

At any z: segment length $= dL = (R-z)\,d\theta$ (2)

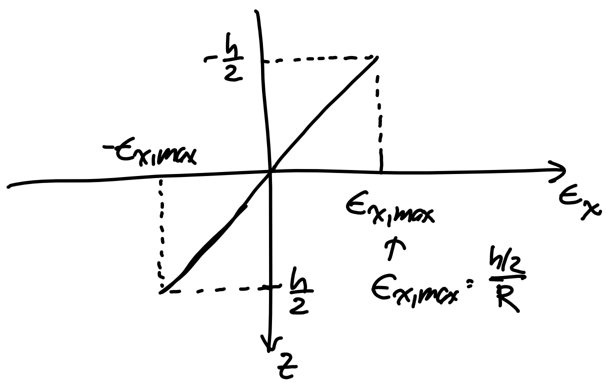
Combine (1) & (2): $dL = dx - z\,d\theta = dx - \frac{z}{R}dx$

Thus, the axial strain @ z:

$$\epsilon_x = \frac{dL-dx}{dx} = -\frac{z}{R}$$

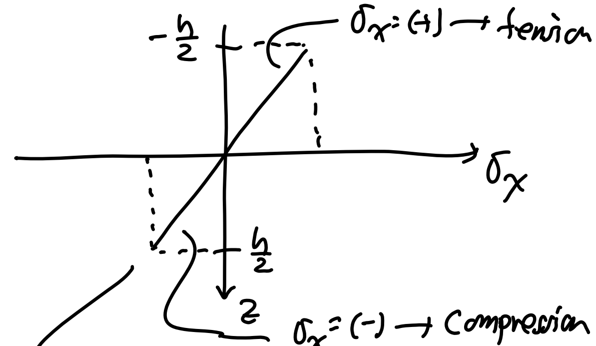
$$\boxed{\epsilon_x = -\frac{z}{R}}$$

Thus, the strain varies linearly along the beam thickness:



Of course, there is a corresponding axial stress:

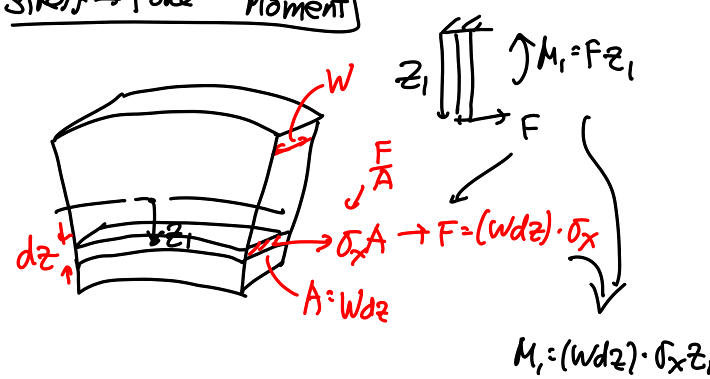
$$\sigma_x = \epsilon_x E = \boxed{-\frac{zE}{R} = \sigma_x}$$



This gradient of stress then generates an internal bending moment!

In response to the original applied moment (from F)

Stress → Force → Moment



⇒ integrate stress through the thickness of the beam:

$$M = \int_{-\frac{h}{2}}^{\frac{h}{2}} [\underbrace{(W dz)\sigma_x}_{\text{force}}] \cdot \underset{\substack{\uparrow \\ \text{moment arm}}}{z}$$

$$= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{EWz^2}{R} dz \Rightarrow \boxed{M = -\left(\frac{1}{12}Wh^3\right)\frac{E}{R}}$$

$$\left[\sigma_x = -\frac{zE}{R}\right]$$

$$\frac{1}{12}Wh^3 = I \triangleq \text{Moment of Inertia}$$

$$\boxed{\frac{1}{R} = -\frac{M}{EI}}$$

Note: (+) radius of curvature

(−) internal bending moment