Lecture 11: Stress Gradients & Beam Combos

- Announcements:
- HW#3 online and due Wednesday morning (next week; this is a one week homework)
- Module 8 on Microstructural Elements online
- Again, this will be a 2 hour lecture
- Midterm is coming in a few weeks: Thursday, March 19
- -----------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Cantilever with residual stress gradient
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients for folded-beam devices
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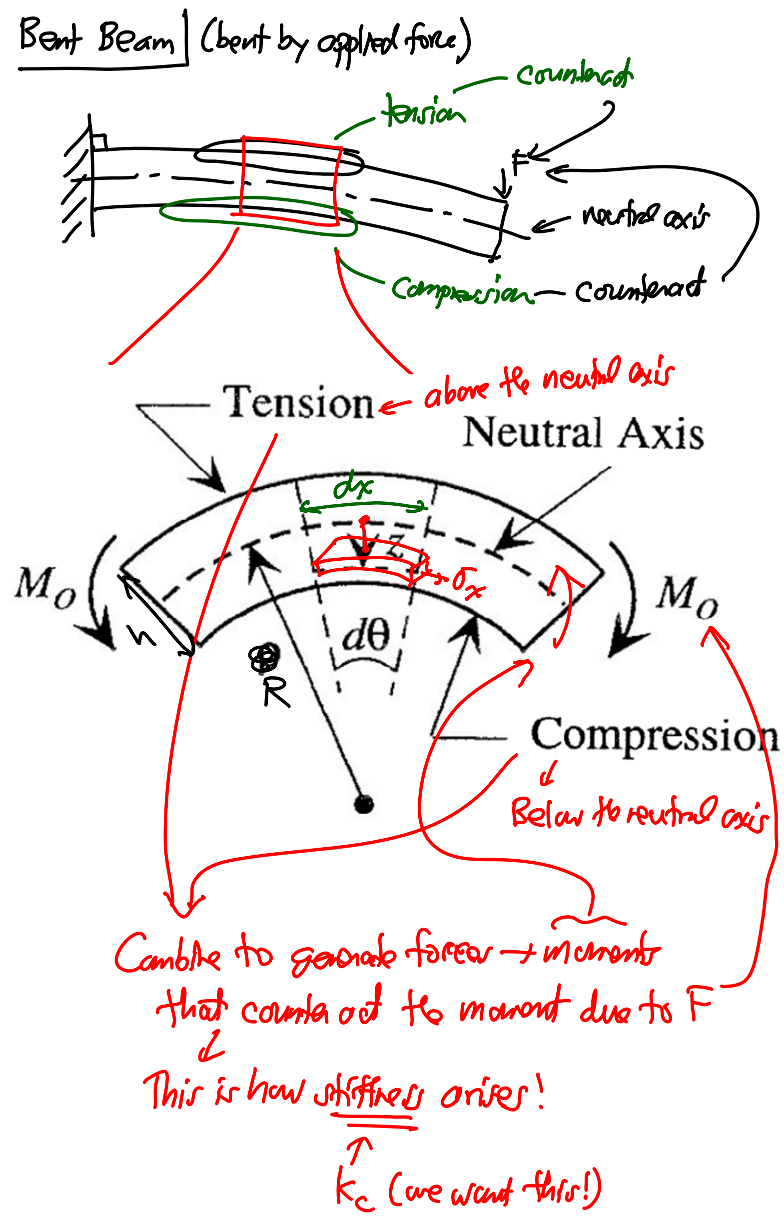
- Last Time:
- Derived governing expression for beam bending



Beam Segment in Pure Bending

⇒ Consider the segment bounded by the dashed lines defining $d\theta$

At $z=0$: neutral axis → segment length $= dx = R\,d\theta$ (1)

At any z: segment length $= dL = (R-z)\,d\theta$ (2)

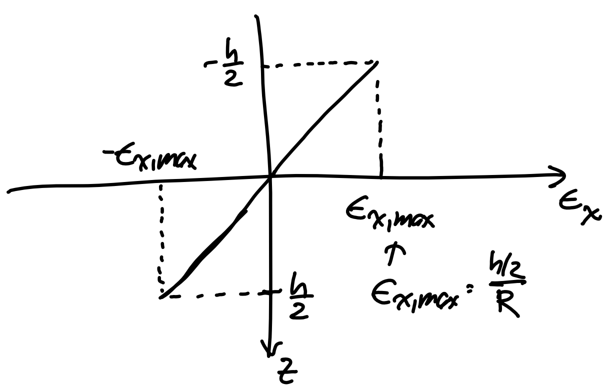
Combine (1) & (2): $dL = dx - z\,d\theta = dx - \frac{z}{R}dx$

Thus, the axial strain @ z:

$$\epsilon_x = \frac{dL-dx}{dx} = -\frac{z}{R}$$

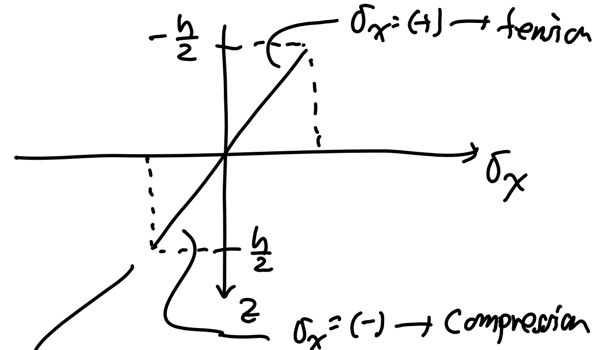
$$\boxed{\epsilon_x = -\frac{z}{R}}$$

Thus, the strain varies linearly along the beam thickness:



Of course, there is a corresponding axial stress:

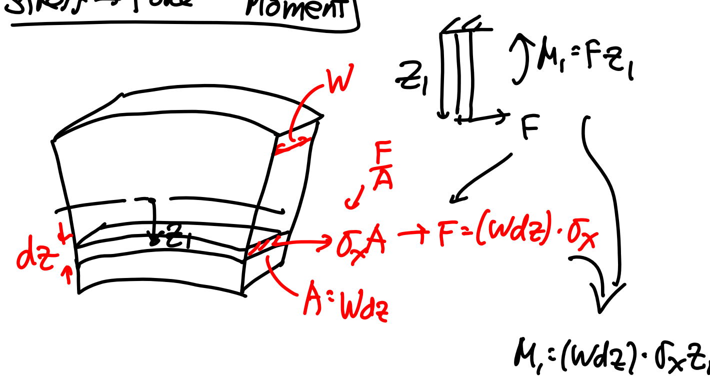
$$\sigma_x = \epsilon_x E = \boxed{-\frac{zE}{R} = \sigma_x}$$



This gradient of stress then generates an internal bending moment!

In response to the original applied moment (from F)

Stress → Force → Moment



⇒ integrate stress through the thickness of the beam:

$$M = \int_{-\frac{h}{2}}^{\frac{h}{2}} \underbrace{[(Wdz)\sigma_x]}_{\text{force}} \cdot \underset{\uparrow \text{ moment arm}}{z}$$

$$= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{EWz^2}{R} dz \Rightarrow \boxed{M = -\left(\frac{1}{12}Wh^3\right)\frac{E}{R}}$$

$$\left[\sigma_x = -\frac{zE}{R}\right]$$

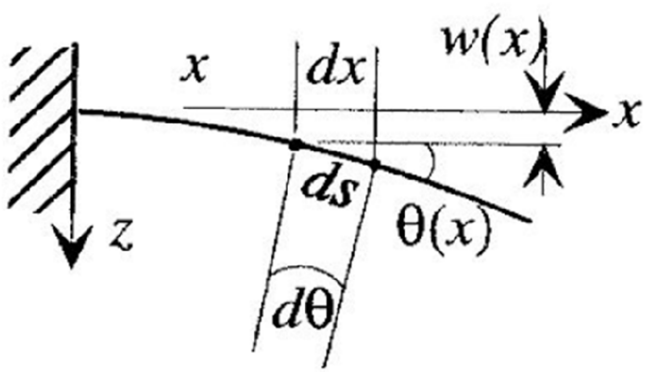
$\frac{1}{12}Wh^3 = I \triangleq$ Moment of Inertia

$$\boxed{\frac{1}{R} = -\frac{M}{EI}}$$

Note: (+) radius of curvature

(−) internal bending moment

<u>Differential Equation for Beam Bending</u>



Write out some geometric relationships:

⇒ then use small angle approx:

$$\cos\theta = \frac{dx}{ds} \rightarrow ds = \frac{dx}{\cos\theta} \longrightarrow ds \simeq dx$$

$$\tan\theta = \frac{dw}{dx} = \text{slope of the beam @ this point} \rightarrow \theta \simeq \frac{dw}{dx} \quad (1)$$

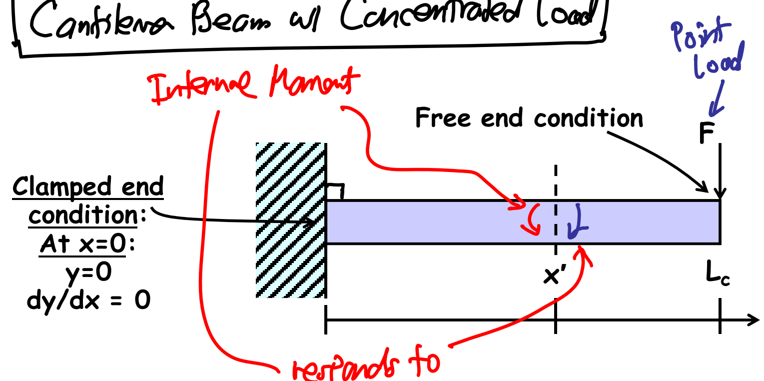
$$ds = R\,d\theta \rightarrow \frac{1}{R} = \frac{d\theta}{ds} \longrightarrow \frac{1}{R} = \frac{d\theta}{dx} \quad (2)$$

Inserting (1) into (2)

$$\frac{1}{R} = \boxed{\frac{d^2w}{dx^2} = -\frac{M}{EI}}$$

← Diff. Eqn. for Small Angle Beam Bending

<u>Cantilever Beam w/ Concentrated Load</u>



Internal Moment @ position x: $M = -F(L-x)$

Thus: $\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$

w/ Clamped End B.C.: $w(x=0)=0,\ \frac{dw}{dx}(x=0)=0$

Free End B.C.: none

Solve for w:

⇒ use Laplace; or a trial solution:

$w = A + Bx + Cx^2 + Dx^3$, then apply B.C.'s

$$\boxed{w = \frac{FL}{2EI}x^2\left(1-\frac{x}{3L}\right)}$$

Deflection @ x due to a point load F applied @ $x=L$

Maximum Deflection → occurs @ $x=L$:

$$w_{max} = \left(\frac{L^3}{3EI}\right)F \rightarrow F = \underbrace{\left(\frac{3EI}{L^3}\right)w(x=L)}_{k_c w(x=L)}$$

stiffness $\triangleq k_c$ @ $x=L$

where $\boxed{k_c = \frac{3EI}{L^3}}$ @ $x=L$

In general, stiffness is a function of location

$$\left[I = \frac{1}{12}Wh^3\right] \Rightarrow \boxed{k_c = \frac{1}{4}EW\frac{h^3}{L^3}}$$

Ex. $L = 100\mu m,\ W = 2\mu m,\ h = 2\mu m$

polysilicon → $E = 150$ GPa

$$k_c = \frac{1}{4}(150G)(2\mu)\left(\frac{2\mu}{100\mu}\right)^3 = 0.6 N/m$$

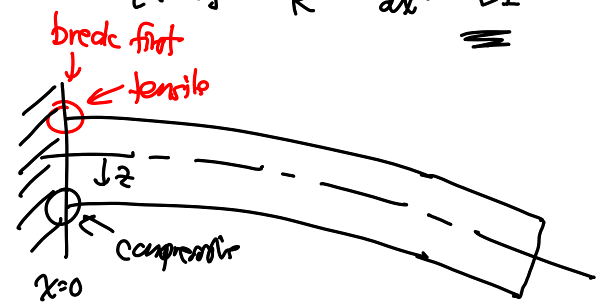
Maximum Stress in a Bent Cantilever

From before, the radius of curvature is given by

$$\frac{1}{R} = \frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

⇒ $\frac{1}{R}$ is maximized (i.e., R is minimized) when $x=0$:

$$[x=0] \Rightarrow \frac{1}{R} = \frac{d^2w}{dx^2} = \frac{FL}{EI}$$



Strain is maximized:

① At the top surface → tensile

② At the bottom surface → compressive

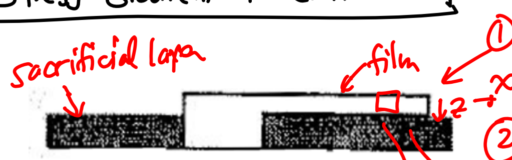
$$\epsilon_{max} = \frac{z}{R} = \frac{h}{2}\frac{1}{R} = \boxed{\frac{h}{2}\frac{FL}{EI} = \epsilon_{max}}$$

$$\left[I = \tfrac{1}{12} W h^3\right] \Rightarrow \epsilon_{max} = \frac{\cancel{1}}{2}\frac{FL}{E}\left(\frac{12}{Wh^{\cancel{3}2}}\right) = \frac{6L}{EWh^2}F$$

$$\boxed{\sigma_{max} = \epsilon_{max} E = \frac{6L}{Wh^2}F}$$

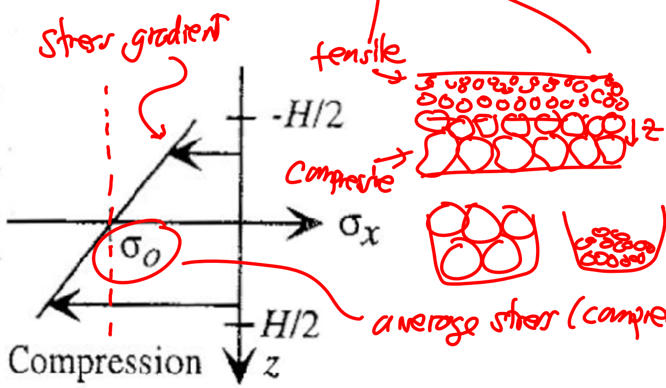
(maximum stress in a bent cantilever subjected to a force F at its tip)

Stress Gradients in Cantilevers



① Deposit film @ high temp.
② Cool it down.

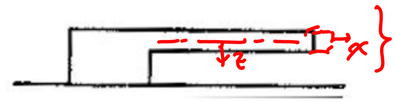
Before release



average stress (compressive)

Stress before release

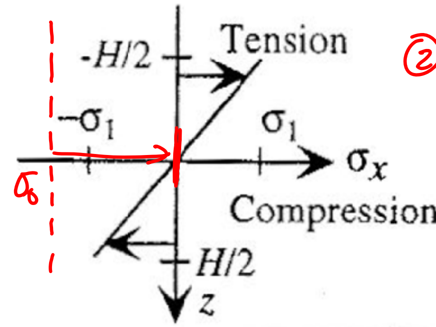
Now, release...



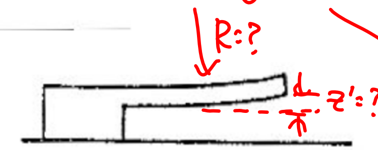
} Intermediate State

After release, but before bending

① Remove the sac. layer & beam free to do what it wants!
② Beam stretches + removes average axial stress
③ Bends to relieve stress gradient

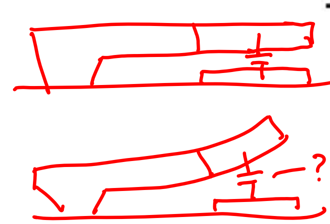


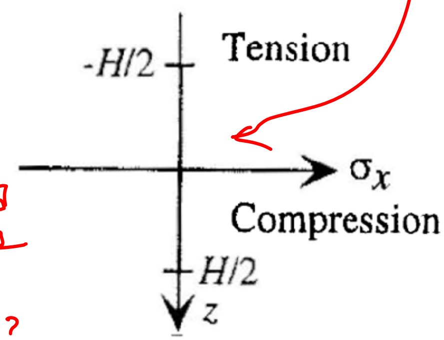
Stress after relea... but before bend



After bending

All stress relieved (state of minimum energy)





After bending

Bending Due to Stress Gradients

Find the radius of curvature:

Prior to release, axial stress: $\sigma = \sigma_0 - \frac{\sigma_1}{(H/2)} z$

The internal moment:

$$M_x = \int_{-\frac{H}{2}}^{\frac{H}{2}} \left[(W \cdot dz)\sigma\right] \cdot z = \int_{-\frac{H}{2}}^{\frac{H}{2}} W\left(z\sigma_0 - \frac{\sigma_1 z^2}{(H/2)}\right) dz$$

$$= W\left(\frac{1}{2}\sigma_0 z^2 - \frac{2\sigma_1 z^3}{3H}\right)\Bigg|_{-H/2}^{H/2}$$

$$= W\left(\cancel{\frac{1}{2}\sigma_0\frac{H^2}{4}} - \frac{2}{3}\sigma_1\frac{H^2}{8} - \cancel{\frac{1}{2}\sigma_0\frac{H^2}{4}} - \frac{2}{3}\frac{\sigma_1 H^2}{8}\right)$$

avg stress cancels

$$\boxed{M_x = -\frac{1}{6}\sigma_1 W H^2}$$

Thus, the radius of curvature:

$$\frac{1}{R} = -\frac{M_x}{E'I} \rightarrow R = \frac{E'I}{M_x} = \frac{1}{2}\frac{E'H}{\sigma_1}$$

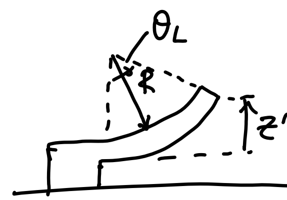
Biaxial Modulus

$[I = \frac{1}{12}WH^3]$

$$\boxed{R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1}}$$

Radius of Curvature for a Cantilever w/ a Stress Gradient

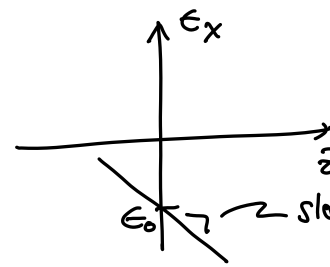
Radius of Curvature → z'



→ Integrate over θ_L to get z'

(do this in homework)

Definition: Strain Gradient

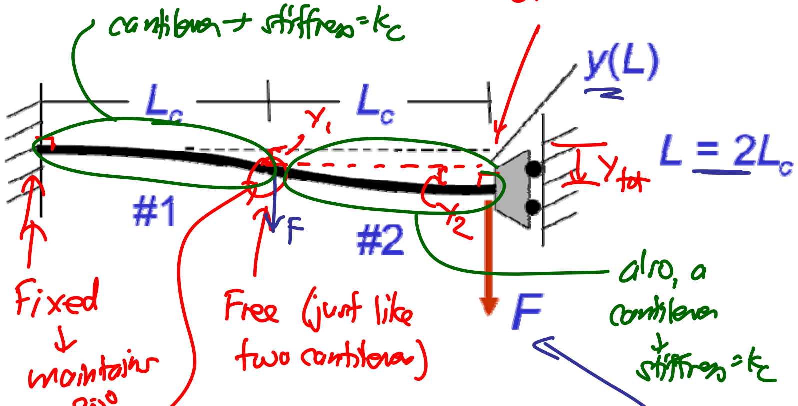


slope $\triangleq$ Strain Gradient $= \Gamma$

$$\Gamma = \frac{\epsilon_1}{(H/2)}$$

$$R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1} = \frac{H}{2}\frac{E'}{\sigma_1} = \frac{(H/2)}{\epsilon_1} = \frac{1}{\Gamma} \rightarrow \boxed{\Gamma = \frac{1}{R}}$$ ✓

Series Combination of Springs



Series: $y_{tot} > y_1$, $y_{tot} > y_2$ $\Big\}$ $y_{tot} = y_1 + y_2$

What's the force here? → F

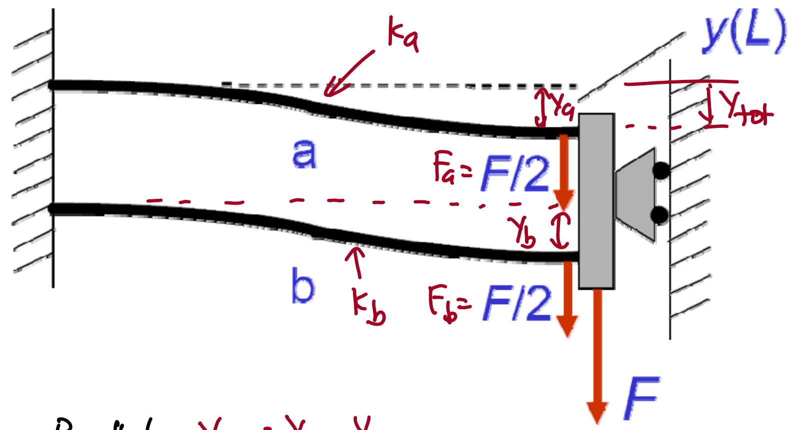
$$y(L) = \frac{F}{k} = 2y(L_c) = 2\left(\frac{F}{k_c}\right) = F\left(\frac{1}{k_c}+\frac{1}{k_c}\right)$$

stiffness of the whole thing

$$\boxed{\frac{1}{k} = \frac{1}{k_c}+\frac{1}{k_c}} \rightarrow \boxed{k = k_c \| k_c}$$

Definition for "||" $\Big\{$ $A\|B = \dfrac{1}{\frac{1}{A}+\frac{1}{B}} = \dfrac{AB}{A+B}$

Parallel Combination of Springs



Parallel: $y_{tot} = y_a = y_b$

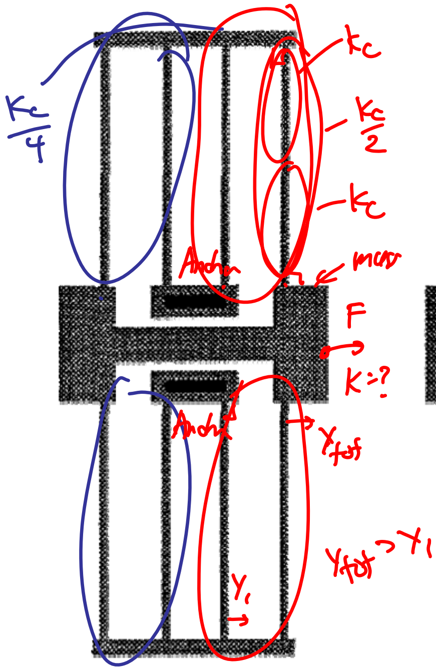
$$y(L) = \frac{F}{k} = \frac{\overset{F/2}{F_a}}{k_a} = \frac{\overset{F/2}{F_b}}{k_b} = \left(\frac{F}{2}\right)\left(\frac{1}{k_a}\right)$$

of the whole thing

$$\boxed{k = 2k_a}$$

In general $\boxed{k_{tot} = k_a + k_b}$

For EE's: springs combine like capacitors



(a) Inner fold, continuous truss

(b) Inner fold, discontinuous truss

$k = 4\left(\frac{k_c}{4}\right) = k_c$