Lecture 12: Beam Combos

- **Announcements**:
- HW#3 online and due Wednesday morning (next week; this is a one week homework)
- Midterm is coming in a few weeks: Thursday, March 19
- Note that this course is using Piazza, so if you have questions about homework problem statements or anything else, you're encouraged to use Piazza
- -------------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Cantilever with residual stress gradient
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients for folded-beam devices
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- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
  - Estimating Resonance Frequency
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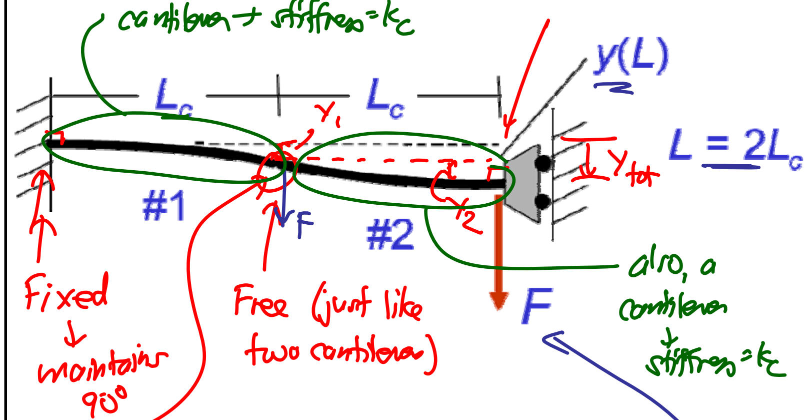
- **Last Time**:
- Spring constant determination
- Going through examples

Series Combination of Springs



Series: $y_{tot} > y_1$, $y_{tot} > y_2$ } $y_{tot} = y_1 + y_2$

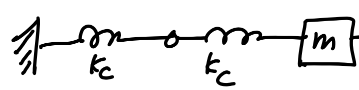
What's the force here? → F

$$y(L) = \frac{F}{k} = 2y(L_c) = 2\left(\frac{F}{k_c}\right) = F\left(\frac{1}{k_c} + \frac{1}{k_c}\right)$$

stiffness of the whole thing

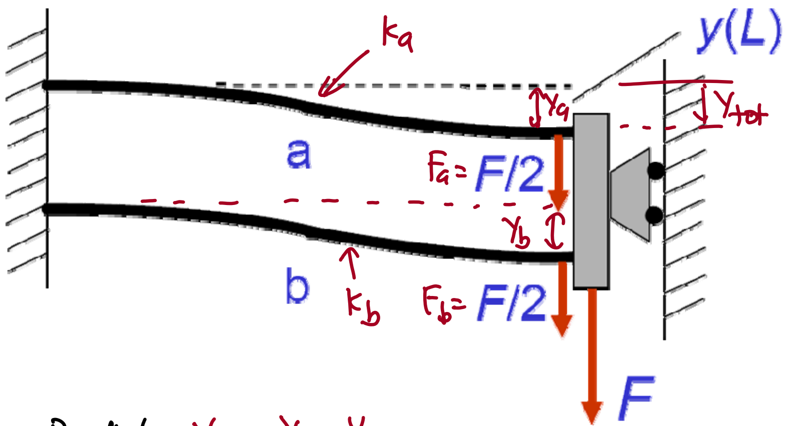
$$\boxed{\frac{1}{k} = \frac{1}{k_c} + \frac{1}{k_c}} \rightarrow \boxed{k = k_c \| k_c}$$

Definition for "‖" $\left\{ A\|B = \frac{1}{\frac{1}{A} + \frac{1}{B}} = \frac{AB}{A+B} \right.$



$F = kx$ ↳ $k = k_c \| k_c$

Parallel Combination of Springs



Parallel: $Y_{tot} = Y_a = Y_b$

$$Y(L) = \frac{F}{k} = \frac{F_a}{k_a} = \frac{F_b}{k_b} = \left(\frac{F}{2}\right)\left(\frac{1}{k_a}\right)$$

$F_a = \frac{F}{2}$, $F_b = \frac{F}{2}$

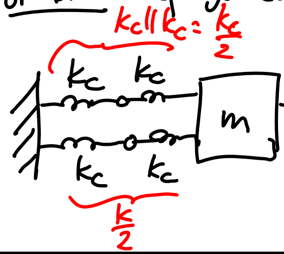
of the whole thing

$\boxed{k = 2k_a}$

In general $\boxed{k_{tot} = k_a + k_b}$

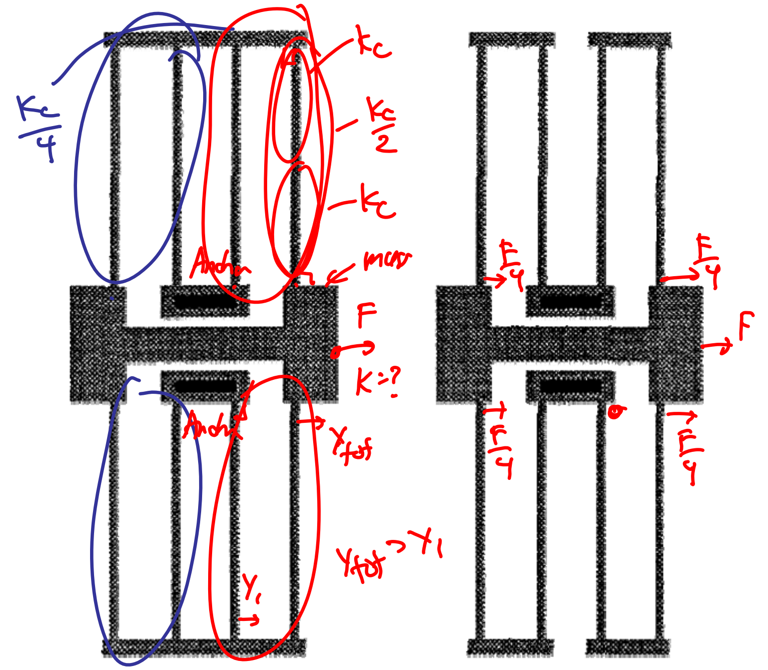
For EE's: springs combine like capacitors

$k_c \| k_c = \frac{k_c}{2}$



$F = kx \rightarrow x = \frac{F}{k}$

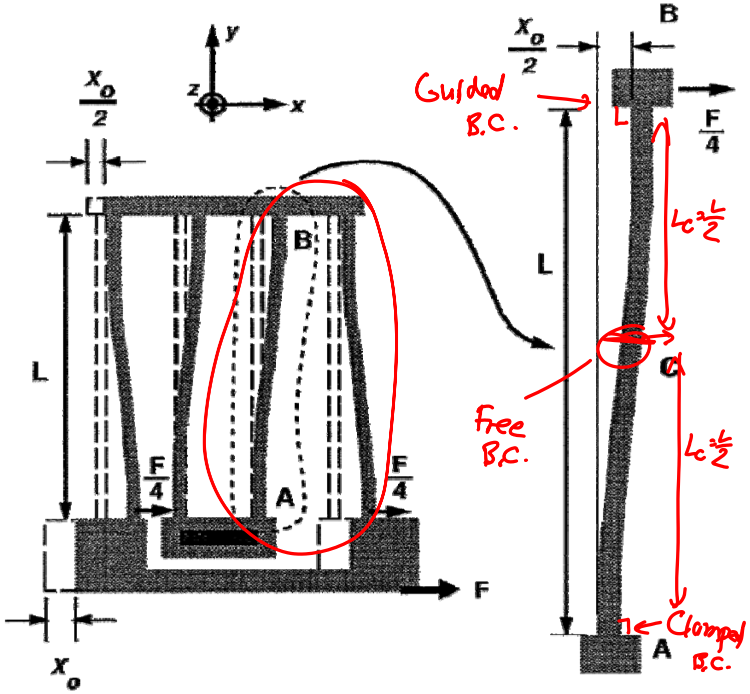
$k = \frac{k_c}{2} + \frac{k_c}{2} = k_c$

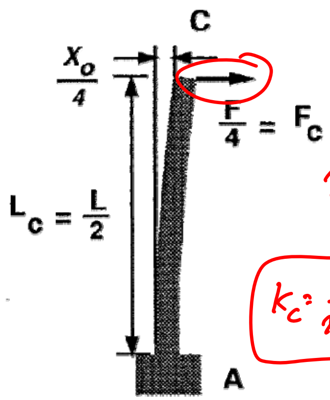


(a) Inner fold, continuous truss

(b) Inner fold, discontinuous truss

$k = 4\left(\frac{k_c}{4}\right) = k_c$





$$x(y) = \frac{F_c L_c}{2EI_z} y^2 \left(1 - \frac{y}{3L_c}\right)$$

$$x(y) = \frac{F_c y^2}{6EI_z}(3L_c - y)$$

thickness

$$I_z = \frac{1}{12} h W^3$$

$$\boxed{k_c = \frac{F_c}{x(L_c)} = \frac{3EI_z}{L_c^3} = \frac{1}{4}Eh\frac{W^3}{L_c^3}}$$

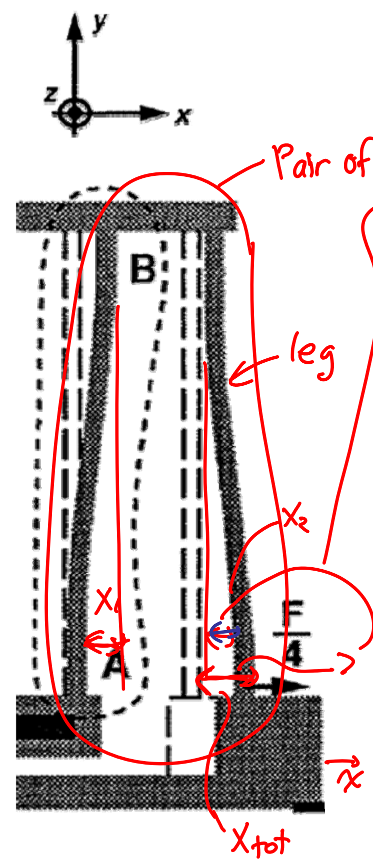
length of cantilever

⇒ insert $L_c = \frac{L}{2}$,

$$\boxed{k_c = \frac{3EI_z}{(L/2)^3} = \frac{24EI_z}{L^3}}$$

stiffness of cantilever of length L_c

full beam length



legs in series, because the total displacement x_{tot} equals the sum of the displacements of the individual legs $x_1 + x_2$

$F_{pair} = \frac{F}{4}$ ← applied to the shuttle

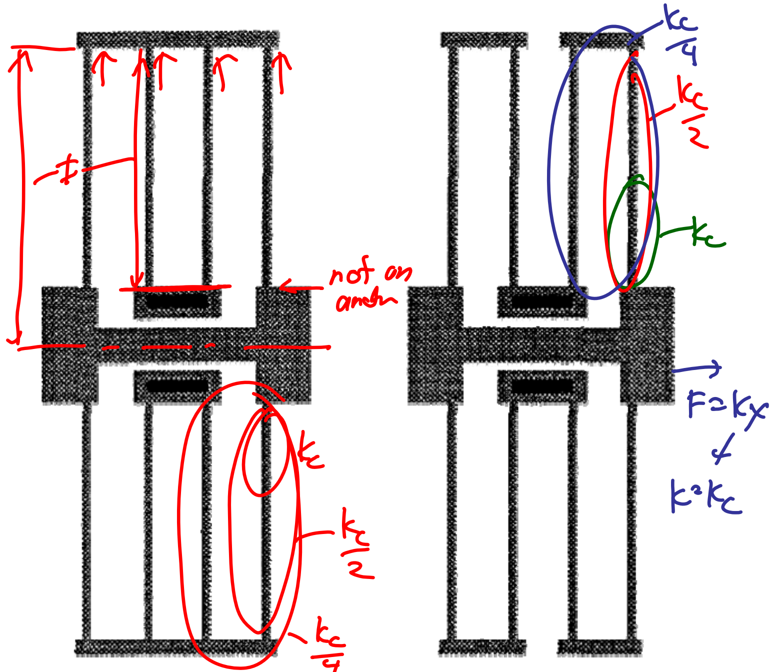
$$x = \frac{F_{pair}}{k_{pair}} = \frac{F_{pair}}{(k_{leg} \| k_{leg})} = \frac{F}{4}\left(\frac{1}{k_{leg}} + \frac{1}{k_{leg}}\right)$$

From before: $k_{leg} = k_c \| k_c = \frac{k_c}{2}$

Thus:

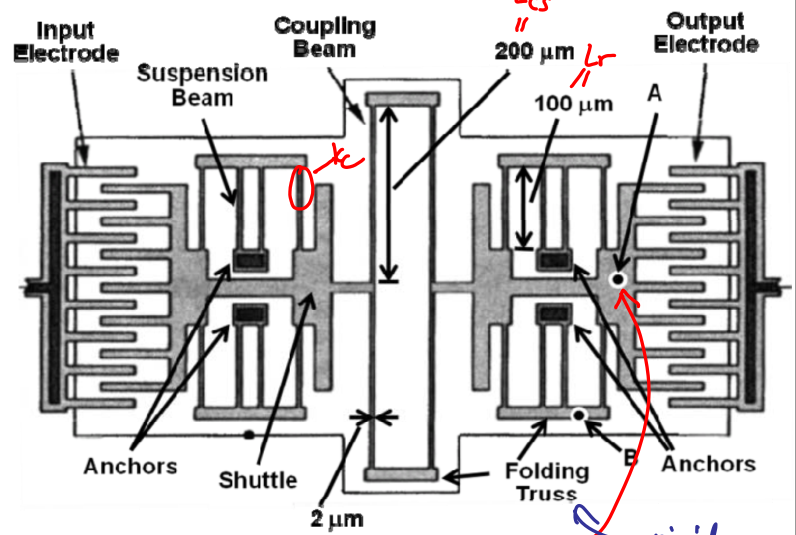
$$x = x_{tot} = \left(\frac{F}{4}\right)\left(\frac{2}{k_c}+\frac{2}{k_c}\right) = \frac{F}{k_c} = \frac{F}{k_{tot}}$$

$$\boxed{k_{tot} = k_c = \frac{24EI_z}{L^3}}$$

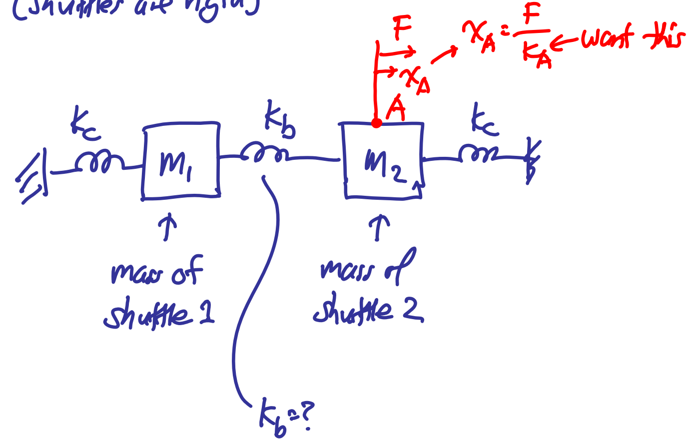


(a) Inner fold, continuous truss
(b) Inner fold, discontinuous truss

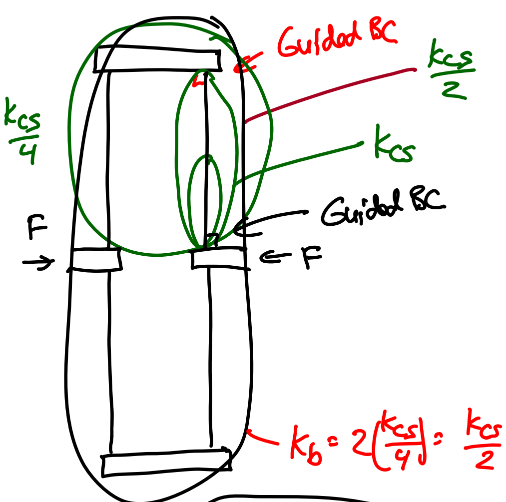
Micromechanical Filter



⇒ Find the stiffness at point A
(shuttles are rigid)



Get k_b:



$k_b = 2\left(\frac{k_{cs}}{4}\right) = \frac{k_{cs}}{2}$

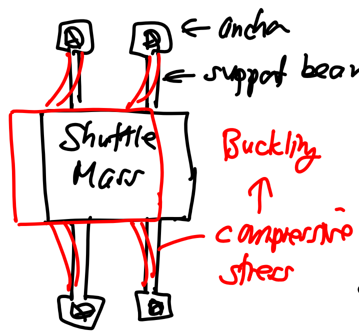
$k_A = k_c + k_c \| k_b = k_c + k_c \| \frac{k_{cs}}{2} = k_A$

where $k_c = \frac{24EI_z}{L_r^3}$

$k_{cs} = \frac{24EI_z}{L_{cs}^3}$

Benefit of Folded-Beam Suspension

⇒ eliminates? stress



① Pursue dep. @ high T → stress free

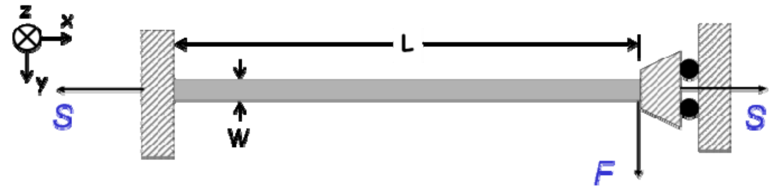
② Cool to RT → stressed

Solution: Folded-Beams

How good a solution?

Tensioned Spring (Non-Ideality)

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L

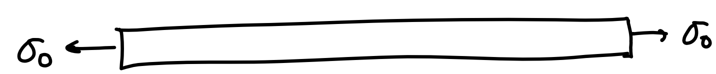


Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4 y}{dx^4} - \underset{\text{Axial Load}}{S} \frac{d^2 y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

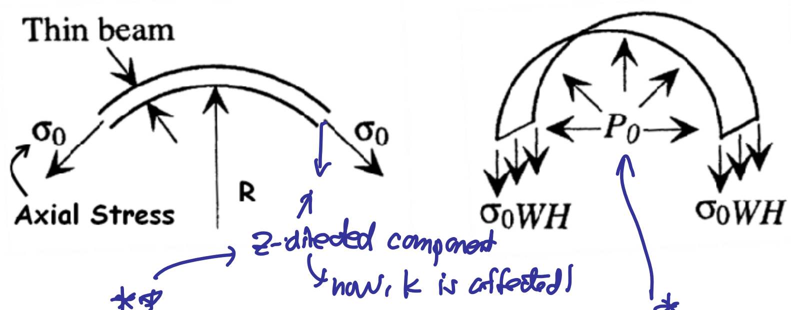
Heuristic Derivation for the Euler Beam Equation

Consider first a straight beam under axial stress: ↓z



⇒ no effect on z-directed stiffness when the beam is straight!

...but when the beam is bent:



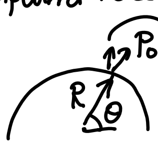
* Upward pressure P_o to counteract the downward force from ** to keep everything in static equilibrium

For ease of analysis:

Assume the beam is bent to an angle π

→ downward vertical force: $2\sigma_o WH$

Upward Force due to P_o:



$P_y(\theta) = P_o \sin\theta$

$$F_u = \int_0^{\pi} (P_o \sin\theta)\, W (R\, d\theta)$$

$$= -P_o W R \cos\theta \Big|_0^{\pi}$$

$$= 2RWP_o$$

[Equilibrium] ⇒ $2RWP_o = 2\sigma_o WH \rightarrow P_o = \dfrac{\sigma_o H}{R}$

$\left[q_o = \dfrac{\text{beam load}}{\text{unit length}} = P_o W \;,\; \dfrac{1}{R} = \dfrac{d^2 w}{dx^2} \right]$ ⇒ w: z-directed beam displ.

$$q_o = \sigma_o WH \frac{d^2 w}{dx^2}$$ ← generalizes to the case of smaller displacements & angles

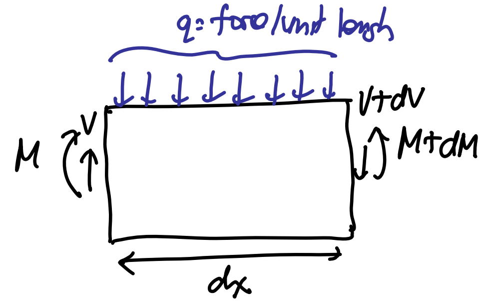
Using the differential beam bending Equation

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \longrightarrow \frac{d^4 w}{dx^4} = \frac{q}{EI}$$

???

q ← $\dfrac{\text{load}}{\text{unit length}}$

* Relationships Between Forces on a Fully Loaded Differential Beam Element



[Total Static Equilibrium] ⇒ total force = 0

F_T = total force: $q\,dx + (\cancel{V} + dV) - \cancel{V} = 0$

$$\therefore \boxed{\frac{dV}{dx} = -q} \qquad (1)$$