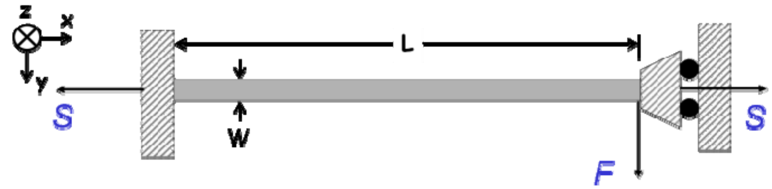
Lecture 13: Beam Combos II & Energy Methods

- Announcements:
- HW#3 due tomorrow morning
- HW#4 online soon; due next Wednesday morning
- Module 9 on Energy Methods is online
- Midterm is next week, Thursday, March 19
- Prof. Nguyen's Wednesday office hours cancelled this week (since I'm giving a tutorial on MEMS-Based Oscillators for BSAC Industry at this time)
- ------------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Cantilever with residual stress gradient
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients for folded-beam devices
- ------------------------------------
- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
  - Estimating Resonance Frequency
- ------------------------------------
- Last Time:

Tensioned Spring (Non-Ideality)

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$



Governing differential equation: (Euler Beam Equation)

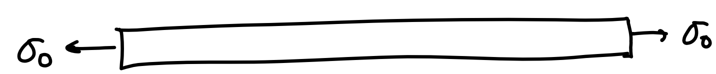
$$EI_z \frac{d^4y}{dx^4} - S\frac{d^2y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ } x=L}$$

Axial Load

Heuristic Derivation for the Euler Beam Equation

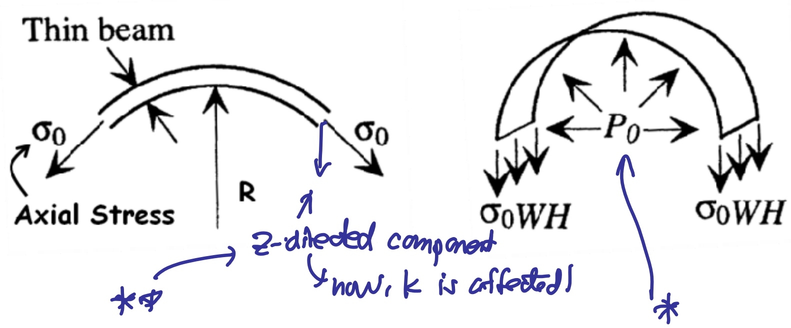
Consider first a straight beam under axial stress:

↓ ~~z~~ y



⇒ no effect on z-directed stiffness when the beam is straight!

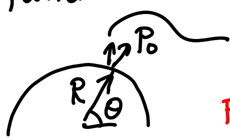
...but when the beam is bent:



* Upward pressure P_o to counteract the downward force from ** to keep everything in static equilibrium

For ease of analysis:
Assume the beam is bent to an angle π
→ downward directed force: $2\sigma_o WH$

Upward Force due to P_o:



$P_y(\theta) = P_o \sin\theta$

$F_u = \int_0^{\pi} (P_o \sin\theta)\, W (R\, d\theta)$

$= -P_o W R \cos\theta \Big|_0^{\pi}$

$= 2RWP_o$

[Equilibrium] ⇒ $2RWP_o = 2\sigma_o WH \rightarrow P_o = \dfrac{\sigma_o H}{R}$

$\left[q_o = \dfrac{\text{beam load}}{\text{unit length}} = P_o W \,, \quad \dfrac{1}{R} = \dfrac{d^2 w}{dx^2} \right]$ ⇒ z-directed beam displ.

$q_o = \sigma_o WH \dfrac{d^2 w}{dx^2}$ ← generalizes to the case of smaller displacements & angles

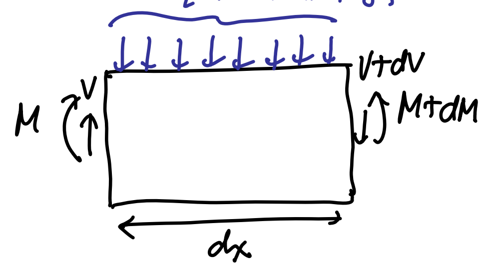
Using the differential beam bending Equation

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \longrightarrow \frac{d^4 w}{dx^4} = \frac{q}{EI}$$

???

q ← load / unit length

* Relationships Between Forces on a Fully Loaded Differential Beam Element



[Total Static Equilibrium] ⇒ total force = 0

F_T = total force = $q\,dx + (\cancel{V} + dV) - \cancel{V} = 0$

$\therefore \boxed{\dfrac{dV}{dx} = -q}$ (1)

⇒ also, total moment w/r to the left hand edge = 0

$M_T = (\cancel{M} + dM) - \cancel{M} - (V + \cancel{dV})dx - \cancel{\tfrac{1}{2} q\, dx^2} = 0$

[neglect products of differentials] $\int_0^{dx} (q\,du)u = \frac{1}{2} q\, dx^2$

$dM - V dx = 0 \longrightarrow \boxed{\dfrac{dM}{dx} = V}$ (2)

Using (1) & (2):

$$\frac{d^2 M}{dx^2} = \frac{dV}{dx} = -q$$

$$EI \frac{d^4 w}{dx^4} = q + q_o$$

q ← external load

q_o ← equiv. load f/ axial stress

$\left[q_0 = \sigma_0 W H \frac{d^2 w}{dx^2}\right] \Rightarrow$

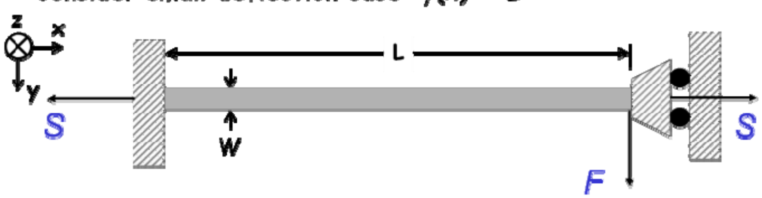
$$EI\frac{d^4 w}{dx^4} - \underbrace{(\sigma_0 W H)}_{\text{tension in the beam} = S}\frac{d^2 w}{dx} = q$$

a force

Euler Beam Equation

Clamped-Guided Beam Under Axial Load

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L



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Axial Load

- Can solve the ODE using standard methods
  - Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)
  - For solution to the clamped-guided case: see S. Timoshenko, Strength of Materials II: Advanced Theory and Problems, McGraw-Hill, New York, 3rd Ed., 1955
- Result from Timoshenko:

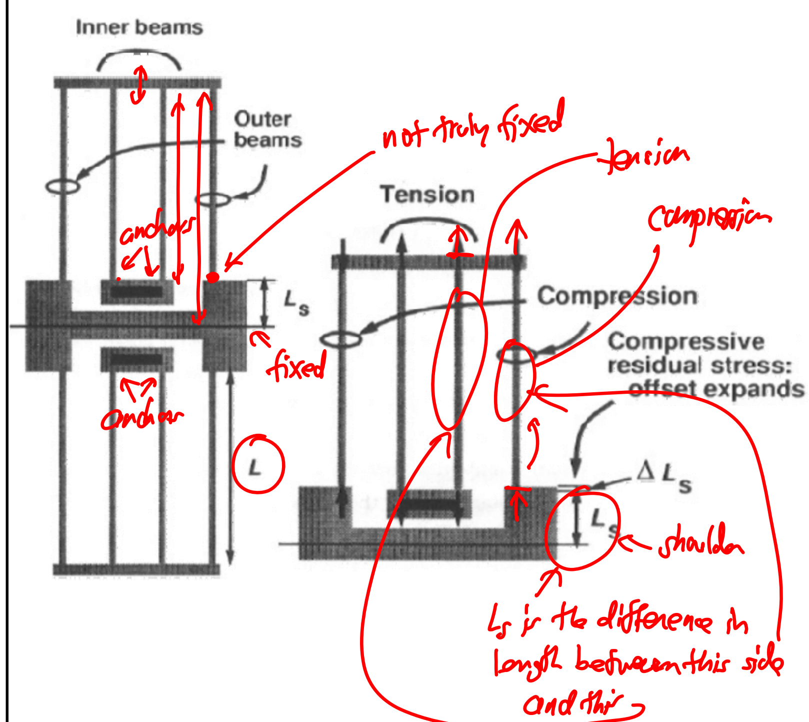
S > 0 (tension) $\quad k^{-1} = \frac{pL - 2\tanh(pL/2)}{p|S|} = \frac{y(x=L)}{F}$

S < 0 (compression)

$$k^{-1} = \frac{-pL + 2\tan(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

force

where $p = \sqrt{\frac{|S|}{EI_z}}$



① If poly-Si strain is ϵ_r, then shoulder expands by $\Delta L_s = \epsilon_r L_s$

② This then applies a load to the beams, $\Delta L = \Delta L_s$

③ Beam Strain & Stress

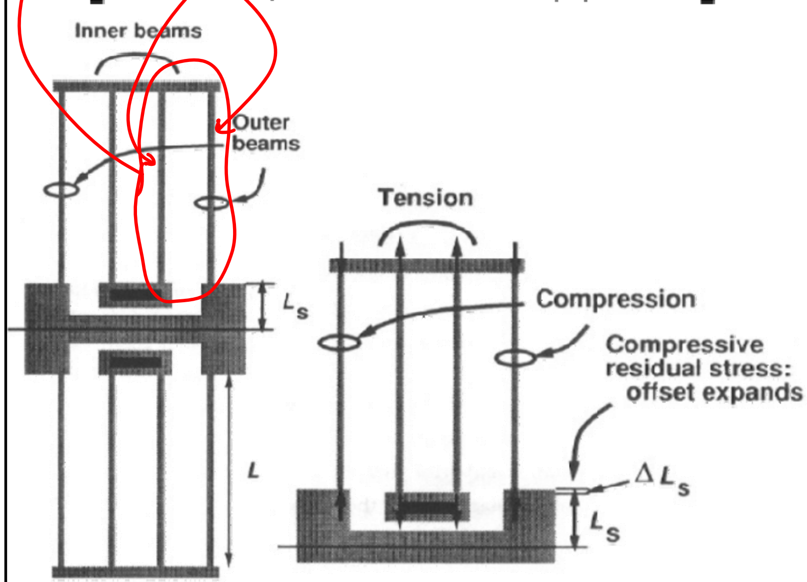
$$\epsilon_b = \frac{\Delta L}{2L} = \frac{\Delta L_s}{2L} = \pm \epsilon_r \frac{L_s}{2L}$$

Stress Force: $S = \pm E \epsilon_r \left(\frac{L_s}{2L}\right) Wh$ (axial tension)

④ Spring Constants:

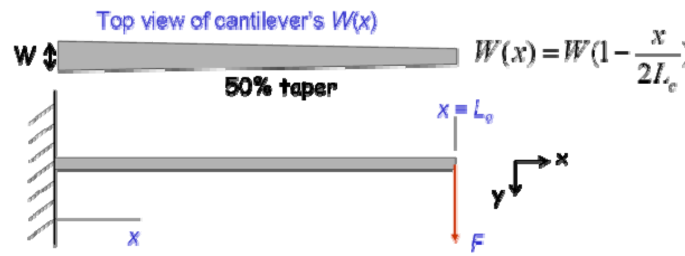
$$k = 4\left(k_{com}^{-1} + k_{ten}^{-1}\right)^{-1}$$

$$k = 4\left[\frac{-pL + 2\tan(pL/2)}{p|S|} + \frac{pL - 2\tanh(pL/2)}{p|S|}\right]^{-1}$$

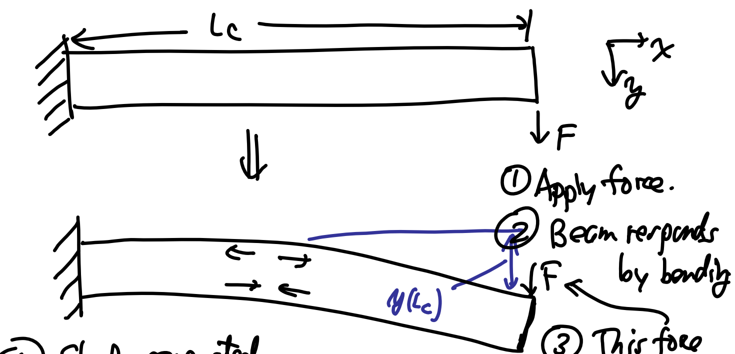


More General Geometries

- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- Example: tapered cantilever beam
- Objective: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)



Same Problem As Before: Take a beam, apply a force.



$W = F \cdot y(L_c)$

④ Strain generated
→ so the beam has received an influx of stored energy
→ magnitude of " " determined by the shape the beam takes!

⑤ Then

U: Stored Energy − Work Done $\xrightarrow[x]{} 0$

$y(x) = f(x)$ { When we choose the right shape!

↓

This is how we get the beam's response to force F.

Fundamentals of Energy Density

General Definition of Work:

$$W(q_1) = \int_0^{q_1} e(q)\,dq \qquad q = \text{displacement} \quad e = \text{effort}$$

for EE: $W(Q) = \int_0^Q \frac{Q}{C} dQ$

Strain Energy Density

$$w = \int_0^{\epsilon_x} \sigma_x \, d\epsilon_x$$

ϵ_x ← value of strain @ position (x,y,z)

$\sigma_x = f(\epsilon_x)$ → relates stress to strain @ position (x,y,z)

$[\sigma_x = E\epsilon_x]$

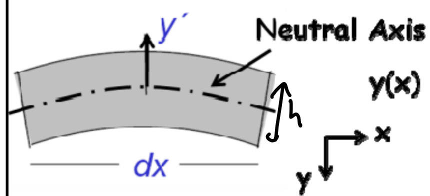
$$w = \int_0^{\epsilon_x} E\epsilon_x \, d\epsilon_x = \frac{1}{2}E\epsilon_x^2$$

Total Strain Energy: [J]

$$W = \iiint \left(\frac{1}{2}E\left(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2\right) + \frac{1}{2}G\left(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2\right)\right) dV$$

dV ← volume

Bending Energy Density



y(x) = transverse displacement of neutral axis

First, find the bending energy dW_{bend} in an infinitesimal length dx:

$$dW_{bend} = W dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\epsilon_x^2(y')\,dy'$$

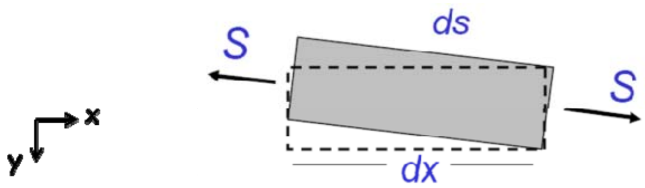
$$\left[\frac{1}{R} = \frac{d^2y}{dx^2},\ \epsilon_x = \frac{y'}{R}\right] \Rightarrow \epsilon_x(y') = y'\frac{d^2y}{dx^2}$$

$$dW_{bend} = W dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\left[y'\frac{d^2y}{dx^2}\right]^2 dy'$$

$$= \frac{1}{2}E\underbrace{\left(\frac{Wh^3}{12}\right)}_{I_z}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\therefore \boxed{W_{bend} = \frac{1}{2}EI_z \int_0^L \left(\frac{d^2y}{dx^2}\right) dx}$$

Energy Due to Axial Load



$\Rightarrow$ energy due to lengthening:

$$ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

[binomial theorem] $\approx dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$

$$\therefore \epsilon_x = \frac{ds - dx}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

$$\left[dW_{axial} = S\epsilon_x dx = \frac{1}{2} S \left(\frac{dy}{dx}\right)^2 dx\right]$$

$$\Downarrow$$

$$\boxed{W_{axial} = \frac{1}{2} S \int_0^L \left(\frac{dy}{dx}\right)^2 dx}$$

Axial Strain Energy

- Go to Module 9, pg. 10, and look at shear strain energy
- Then, finish off Module 9