Lecture 14: Resonance Frequency

- **Announcements:**
- HW#4 online; due next Wednesday morning
- Module 10 on Resonance Frequency is online
- Midterm is next week, Thursday, March 19
- Will pass out and discuss old exams and an Info Sheet on the midterm
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- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
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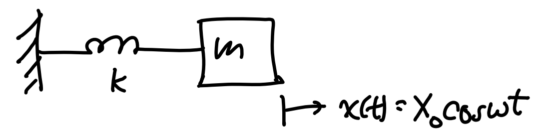
- **Last Time:**
- Finished solution by energy methods
- Now, start Resonance Frequency

Estimating Resonance Frequency



Potential Energy:

$$W(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kX_0^2\cos^2\omega_0 t$$

Kinetic Energy:

$$K(t) = \frac{1}{2}M\dot{x}^2(t) = \frac{1}{2}MX_0^2\omega^2\sin^2\omega t$$

$\dot{x} = \frac{dx}{dt}$ = velocity

Remarks.

① Energy must be conserved.

② Total Energy = Potential Energy + Kinetic Energy at all times & locations on the structure

$$W_{max} = \frac{1}{2}kX_0^2 = K_{max} = \frac{1}{2}M\omega^2X_0^2$$

maximum potential energy

peak displacement

maximum kinetic energy

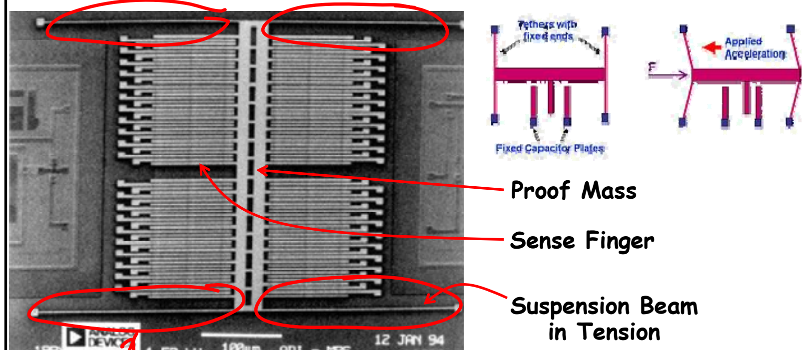
radian frequency

*

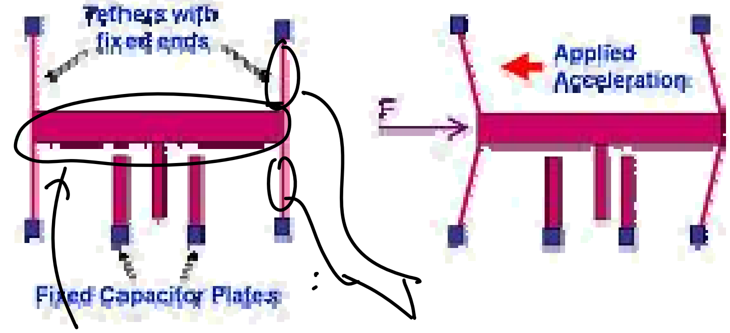
$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$ ⇒ good for problems where the mass & stiffness can be separated, i.e., they're distinct

Ex. ADXL-50 Accelerometer

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - ↳ Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm



In fabrication, purposely introduce a tensile stress in the beams! a large one!

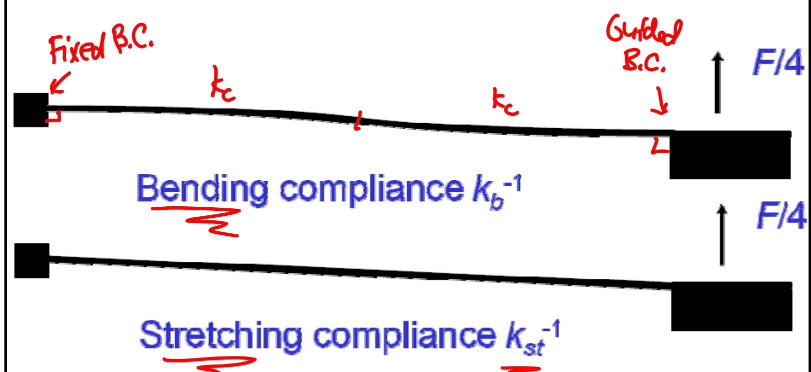


mass of the structure >> mass of springs
∴ ignore the mass of the springs

stiffness of springs << stiffness of structure
∴ ignore the stiffness of the structure

For the ADXL-50: 60% of the mass comes from sense fingers → M = 162 ng

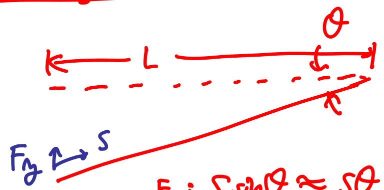
Suspension: Four tensioned beams



Bending Contribution

$$k_b^{-1} = \left(\frac{1}{k_c}+\frac{1}{k_c}\right) = 2\left(\frac{(L/2)^3}{3E(hw^3/12)}\right) = \frac{L^3}{Ehw^3} = 4.2\mu m/\mu N$$

Stretching Contribution



$$F_y = S\sin\theta \approx S\theta \approx S\left(\frac{y}{L}\right) = \underbrace{\left(\frac{S}{L}\right)}_{k_{st}} y$$

$$k_{st}^{-1} = \frac{L}{S} = \frac{L}{\sigma_r Wh} = 1.14\mu m/\mu N$$

To get the total spring constant ← add the bending stiffness to the stretching:

$$k = 4(k_b + k_{st}) = 4(0.24 + 0.88) = 4.5\mu N/\mu m = \underbrace{4.5}_{4.48} N/m$$

Now, get resonance freq.:

$$f = \frac{1}{2\pi}\sqrt{\frac{K}{M}} = \frac{1}{2\pi}\sqrt{\frac{4.48\,N/m}{162\times10^{-12}\,kg}} = 26.5\,kHz$$

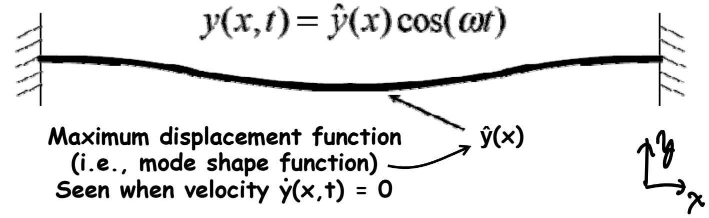
ADXL-50 Data Sheet: $f_o = 24\,kHz$ ↔ difference?

capacitive transduction & electrical stiffness

Find Resonance Frequency When Mass & Stiffness is Distributed

- Vibrating structure displacement function:

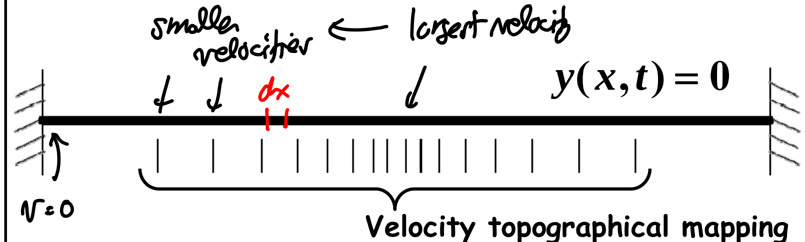
$$y(x,t) = \hat{y}(x)\cos(\omega t)$$



- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy W_{max} at the point of maximum displacement (e.g., when t=0, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

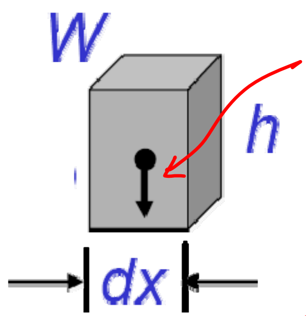
Get Maximum Kinetic Energy

velocity: $v(x,t) = \frac{\partial y(x,t)}{\partial t} = -\omega\hat{y}(x)\sin\omega t$



When $y(x,t) = 0$, all the energy in the structure is kinetic. ($\mathcal{W} = 0$)

$$v\left(x, \frac{(2m+1)\pi}{2\omega}\right) = -\omega\hat{y}(x) \qquad t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \ldots$$



velocity:

$v = -\omega \hat{y}(x)$ $\qquad \frac{(2m+1)\pi}{2W}$

$dK = \frac{1}{2} \cdot dm \cdot [v(x,t)]^2$

$dm = \rho(Whdx)$

density

Maximum K.E.:

$$K_{max} = \int_0^L \frac{1}{2}\rho Wh\,dx\,v^2(x,t) = \int_0^L \frac{1}{2}\rho Wh\omega^2[\hat{y}(x)]^2 dx$$

To get frequency:

$K_{max} = W_{max}$

$$\boxed{\omega = \sqrt{\frac{W_{max}}{\int_0^L \frac{1}{2}\rho Wh[\hat{y}(x)]^2 dx}}}$$ [radians/s]

ω = radian resonance freq.

W_{max}: maximum potential energy

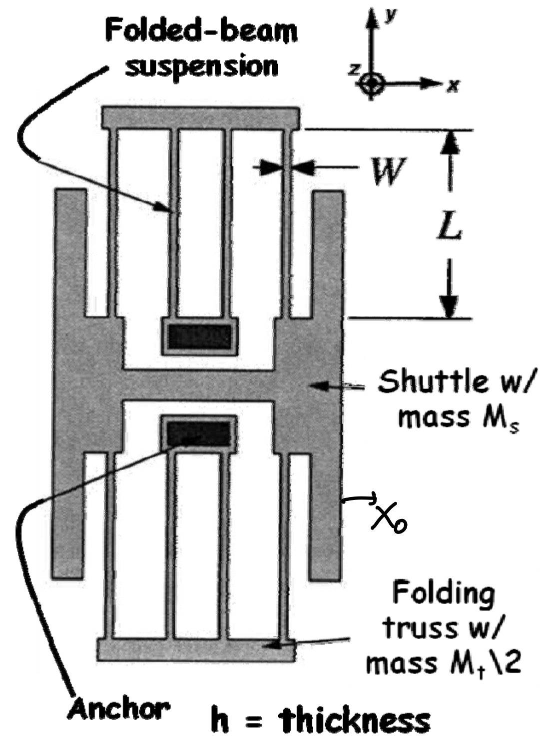
ρ = density of the structural material

W = beam width

h = " thickness

$\hat{y}(x)$ = resonance mode shape

Resonance Freq. of a Folded-Beam Resonator



- Derive an expression for the resonance frequency of the above structure

Use the Rayleigh-Ritz Method: (energy method)

$K_{max} = W_{max}$

Find the kinetic energy → one piece at a time

$$K_{max} = \underbrace{K_s}_{\text{shuttle}} + \underbrace{K_t}_{\text{trusses}} + \underbrace{K_b}_{\text{beams}}$$

$$K_{max} = \frac{1}{2} v_s^2 M_s + \frac{1}{2} v_t^2 M_t + \frac{1}{2}\int v_b^2 \, dM_b$$

mass of both trusses

Velocity of Shuttle: $v_s = \omega_o X_o$

resonance freq.

maximum displacement of the shuttle

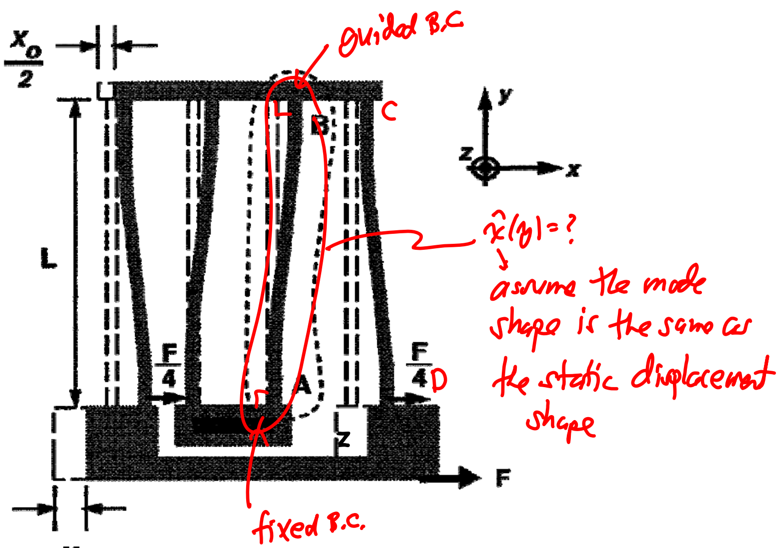
$$\therefore K_s = \frac{1}{2} v_s^2 M_s = \frac{1}{2}\omega_o^2 X_o^2 M_s$$

Velocity of Truss: $v_t = \frac{1}{2} v_s = \frac{1}{2}\omega_o X_o$

$$\therefore K_t = \frac{1}{2}\left(\frac{1}{2}\omega_o X_o\right)^2 M_t = \frac{1}{8}\omega_o^2 X_o^2 M_t$$

mass of both trusses

Velocity of the Beam Segments:



Segment [AB]:

$$\hat{x}(y) = \frac{F_x}{48EI_z}\left(3Ly^2 - 2y^3\right), \quad 0 \le y \le L \qquad (1)$$

At $y = L$: $x(L) = \frac{X_o}{2} = \frac{F_x L^3}{48EI_z}$ ← B.C.

Substitute into (1):

$$\hat{x}(y) = \frac{X_o}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity:

$$v_b(y)\big|_{[AB]} = \frac{X_o}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_o$$

Plugging into the expression for K_b:

$$K_{[AB]} = \frac{1}{2}\int_0^L \frac{X_o^2\omega_o^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

$$= \frac{X_o^2\omega_o^2 M_{[AB]}}{8L}\int_0^L \left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

$M_{[AB]}$: static mass

mass per unit length

$$K_{[AB]} = \frac{13}{280} X_o^2 \omega_o^2 M_{[AB]}$$