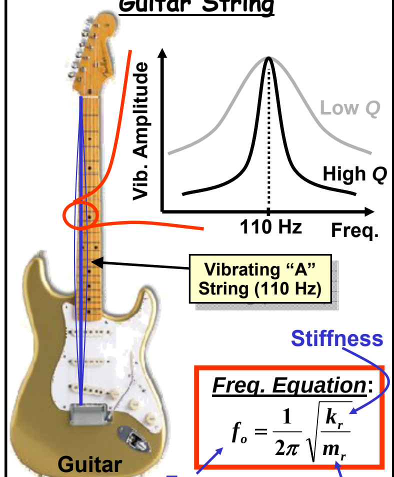
Brute Force Methods for Resonance Frequency Determination

Basic Concept: Scaling Guitar Strings

Guitar String



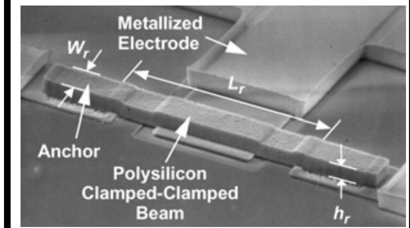
Vibrating "A" String (110 Hz)

Freq. Equation:

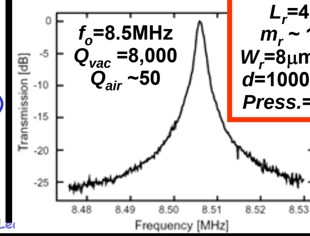
$$f_o = \frac{1}{2\pi}\sqrt{\frac{k_r}{m_r}}$$

Stiffness
Freq.
Mass

μMechanical Resonator

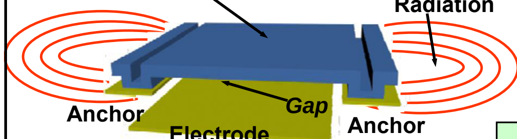


[Bannon 1996]



Performance:
L_r=40.8μm
m_r ~ 10^{-13} kg
W_r=8μm, h_r=2μm
d=1000Å, V_P=5V
Press.=70mTorr

Anchor Losses

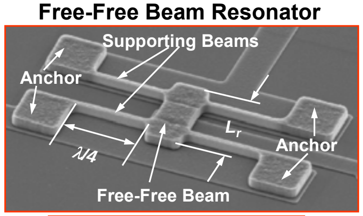


Q = 300 at 70MHz

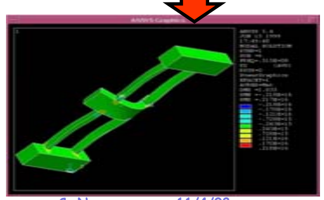
Problem: direct anchoring to the substrate ⇨ anchor radiation into the substrate ⇨ lower Q

Solution: support at motionless nodal points ⇨ isolate resonator from anchors ⇨ less energy loss ⇨ higher Q

Free-Free Beam Resonator

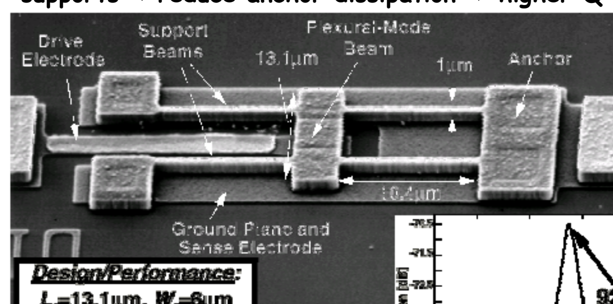


Q = 15,000 at 92MHz



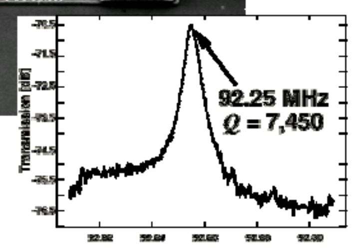
92 MHz Free-Free Beam μResonator

- Free-free beam μmechanical resonator with non-intrusive supports ⇨ reduce anchor dissipation ⇨ higher Q

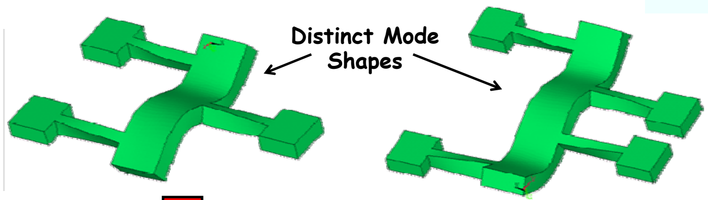


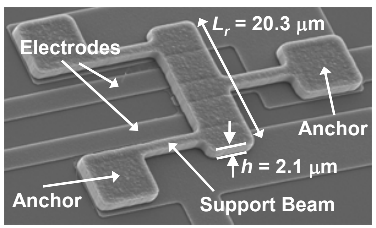
Design/Performance:
L_r=13.1μm, W_r=8μm
h=2μm, d=1000Å
V_P=28-76V, W_e=2.8μm
f_o~92.25MHz
Q~7,450 @ 10mTorr

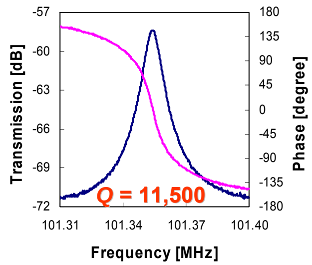
[Wang, Yu, Nguyen 1998]



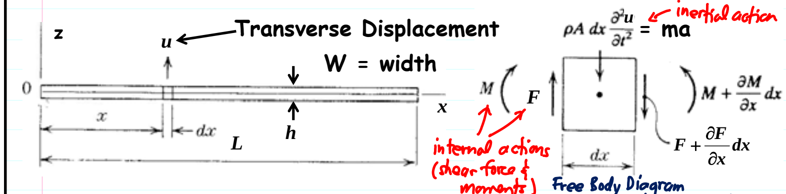
Higher Order Modes for Higher Freq.







Flexural-Mode Beam Wave Equation



- Derive the wave equation for transverse vibration:

Dynamic Equilibrium Condition for Forces in the y-direction:

$$F - \left(F + \frac{\partial F}{\partial x}dx\right) - \rho A dx \frac{\partial^2 u}{\partial t^2} = 0 \quad (1)$$

neglect the $\frac{\partial F}{\partial x}dx$ term

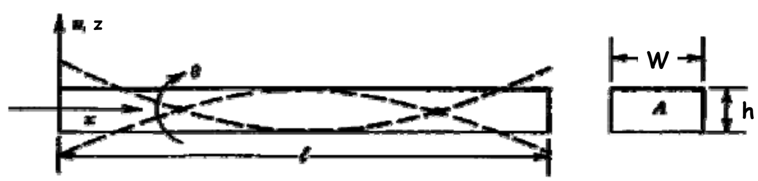
and the moment equilibrium condition: $-F dx + \frac{\partial M}{\partial x} dx \approx 0 \quad (2)$

Combining (1) & (2):

$$\frac{\partial^2 M}{\partial x^2}dx = -\rho A dx \frac{\partial^2 u}{\partial t^2} \rightarrow \frac{\partial^2}{\partial x^2}\left(-EI\frac{\partial^2 u}{\partial x^2}\right) = -\rho A \frac{\partial^2 u}{\partial t^2} \rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = \left(\frac{EI}{\rho A}\right)\frac{\partial^4 u}{\partial x^4}}$$

$$\left[\frac{\partial^2 u}{\partial x^2} = -\frac{M}{EI} \rightarrow M = -EI\frac{\partial^2 u}{\partial x^2}\right] \qquad I_y = \frac{Wh^3}{12}$$

Example: Free-Free Beam



- Determine the resonance frequency of the beam
- Specify the lumped parameter mechanical equivalent circuit
- Transform to a lumped parameter electrical equivalent circuit
- Start with the flexural-mode beam equation:

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{EI}{\rho A}\right)\frac{\partial^4 u}{\partial x^4}$$

Free-Free Beam Frequency

- Substitute $u = u_1 e^{j\omega t}$ into the wave equation:

$$\frac{\partial^4 u}{\partial x^4} = \left(\omega^2 \frac{\rho A}{EI}\right)u \quad (1)$$

- This is a 4th order differential equation with solution:

$$u(x) = \mathcal{A}\cosh kx + \mathcal{B}\sinh kx + \mathcal{C}\cos kx + \mathcal{D}\sin kx \quad (2)$$

Gives the mode shape during resonance vibration.

- Boundary Conditions:

At $x = 0$	At $x = \ell$	
$\frac{\partial^2 u}{\partial x^2} = 0$	$\frac{\partial^2 u}{\partial x^2} = 0$	$M = 0$ (Bending moment)
$\frac{\partial^3 u}{\partial x^3} = 0$	$\frac{\partial^3 u}{\partial x^3} = 0$	$\frac{\partial M}{\partial x} = 0$ (Shearing force)

Free-Free Beam Frequency (cont)

- Applying B.C.'s, get A=C and B=D, and

$$\begin{bmatrix} (\cosh k\ell - \cos k\ell) & (\sinh k\ell - \sin k\ell) \\ (\sinh k\ell + \sin k\ell) & (\cosh k\ell - \cos k\ell) \end{bmatrix} \begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix} = 0 \quad (3)$$

- Setting the determinant = 0 yields

$$\cos k\ell = \frac{1}{\cosh k\ell}$$

- Which has roots at

$$k_1\ell = 4.730 \qquad k_2\ell = 7.853 \qquad k_3\ell = 10.996$$

These values of $k_n\ell$ correspond to the different modes of vibration!

- Substituting (2) into (1) finally yields:

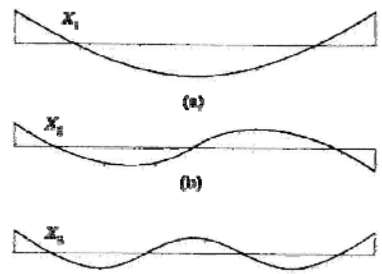
$$k^4 = \frac{\rho A}{EI}\omega^2 \longrightarrow f_n = \frac{(k_n\ell)^2}{2\pi\ell^2}\sqrt{\frac{EI}{\rho A}}$$

[Free-Free Beam Frequency Equation]

Higher Order Free-Free Beam Modes

Mode	n	Nodal Points	$k_n\ell$	f_n/f_1
Fundamental (f_1)	1	2	4.730	1.000
1st Harmonic	2	3	7.853	2.757
2nd Harmonic	3	4	10.996	5.404
3rd Harmonic	4	5	14.137	8.932
4th Harmonic	5	6	17.279	13.344

← More than 10x increase



Fundamental Mode (n=1)

1st Harmonic (n=2)

2nd Harmonic (n=3)

Mode Shape Expression

- The mode shape expression can be obtained by using the fact that A=C and B=D into (2), yielding

$$u_z = \mathcal{B}\left[\left(\frac{\mathcal{A}}{\mathcal{B}}\right)(\cosh kx + \cos kx) + (\sinh kx + \sin kx)\right]$$

- Get the amplitude ratio by expanding (3) [the matrix] and solving, which yields

$$\frac{\mathcal{A}}{\mathcal{B}} = \frac{\sin k\ell - \sinh k\ell}{\cosh k\ell - \cos k\ell}$$

- Then just substitute the roots for each mode to get the expression for mode shape

Fundamental Mode (n=1)
[Substitute $k_1\ell = 4.730$]