Lecture 15: Equivalent Circuits I

- Announcements:
- HW#4 due tomorrow morning; solutions will be released very shortly after
- Module 11 titled Equivalent Circuits I is online
- Midterm this Thursday, March 19
- Passed back HW#3 (extras will be in the box outside my office)
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- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
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- Reading: Senturia, Chpt. 5
- Lecture Topics:
  - Lumped Mechanical Equivalent Circuits
  - Electromechanical Analogies
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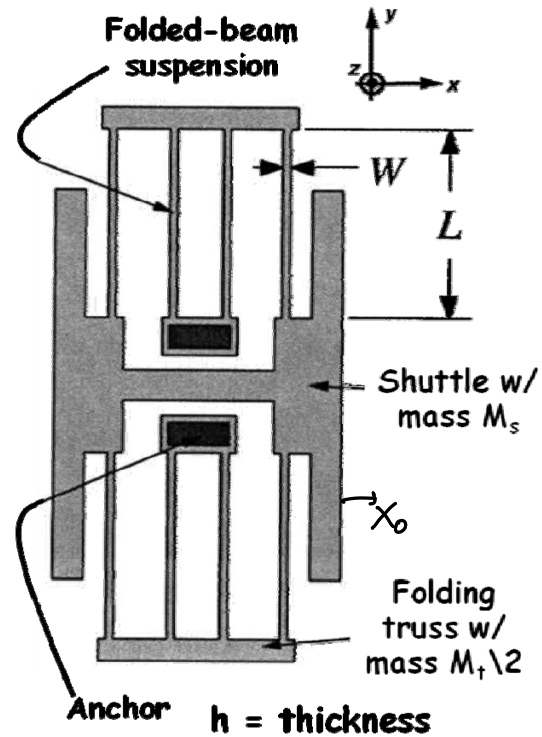
- Last Time:
- Started resonance frequency determination

Resonance Freq. of a Folded-Beam Resonator



- Derive an expression for the resonance frequency of the above structure

Use the Rayleigh-Ritz Method: (energy method)

$$K_{max} = W_{max}$$

Find the kinetic energy → one piece at a time

$$K_{max} = \underbrace{K_s}_{\text{shuttle}} + \underbrace{K_t}_{\text{trusses}} + \underbrace{K_b}_{\text{beams}}$$

$$K_{max} = \frac{1}{2} v_s^2 M_s + \frac{1}{2} v_t^2 M_t + \frac{1}{2}\int v_b^2 \, dM_b$$

(M_t ↑ mass of both trusses)

Velocity of Shuttle: $v_s = \omega_0 X_0$ (ω_0 ↑ resonance freq.; X_0 ↖ maximum displacement of the shuttle)

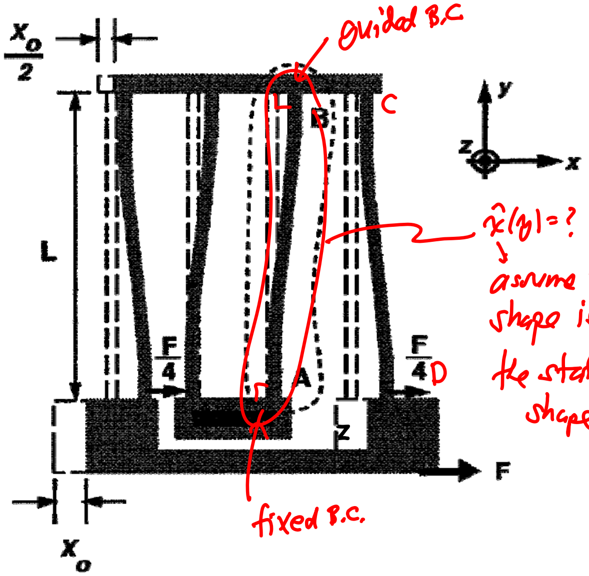
$$\therefore K_s = \frac{1}{2} v_s^2 M_s = \frac{1}{2}\omega_0^2 X_0^2 M_s$$

Velocity of Truss: $v_t = \frac{1}{2} v_s = \frac{1}{2}\omega_0 X_0$

$$\therefore K_t = \frac{1}{2}\left(\frac{1}{2}\omega_0 X_0\right)^2 M_t = \frac{1}{8}\omega_0^2 X_0^2 M_t$$

(M_t ↑ mass of both trusses)

Velocity of the Beam Segments:



assume the mode shape is the same as the static displacement shape

Segment [AB]:

$$\hat{x}(y) = \frac{F_x}{48 E I_z}\left(3Ly^2 - 2y^3\right), \quad 0 \le y \le L \qquad (1)$$

At $y = L$: $x(L) = \frac{X_0}{2} = \frac{F_x L^3}{48 E I_z}$ ← B.C.

Substitute into (1):

$$\hat{x}(y) = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity:

$$v_b(y)\Big|_{[AB]} = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_0$$

Plugging into the expression for K_b:

$$K_{[AB]} = \frac{1}{2}\int_0^L \frac{X_0^2\omega_0^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]} = \frac{X_0^2\omega_0^2 M_{[AB]}}{8L}\int_0^L \left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

$M_{[AB]}$: static mass; $\frac{M_{[AB]}}{L}$ ↑ mass per unit length

$$K_{[AB]} = \frac{13}{280} X_0^2 \omega_0^2 M_{[AB]}$$

$\frac{13}{280}$ ↑ mass modification factor for dynamic mass

For segment [CD]:

$$v_b(y)\Big|_{[CD]} = X_o\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]\omega_o$$

Thus:

$$K_{[CD]} = \frac{X_o^2\omega_o^2 M_{[CD]}}{2L}\int_0^L \left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]^2 dy$$

$$K_{[CD]} = \frac{83}{280} X_o^2 \omega_o^2 M_{[CD]}$$

($\frac{83}{280}$: mass modification factor; $M_{[CD]}$: static mass)

Let $M_b \triangleq$ total mass of all 8 beams

Then: $M_{[AB]} = M_{[CD]} = \frac{1}{8}M_b$

Thus: $K_b = 4K_{[AB]} + 4K_{[CD]} = \frac{6}{35}X_o^2\omega_o^2 M_b$

W_{max} → simply equal to the work done to achieve maximum deflection

$$W_{max} = \frac{1}{2}k_x X_o^2$$

Then, using Rayleigh-Ritz:

$$K_{max} = W_{max}$$

$$X_o^2\omega_o^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right] = \frac{1}{2}k_x X_o^2$$

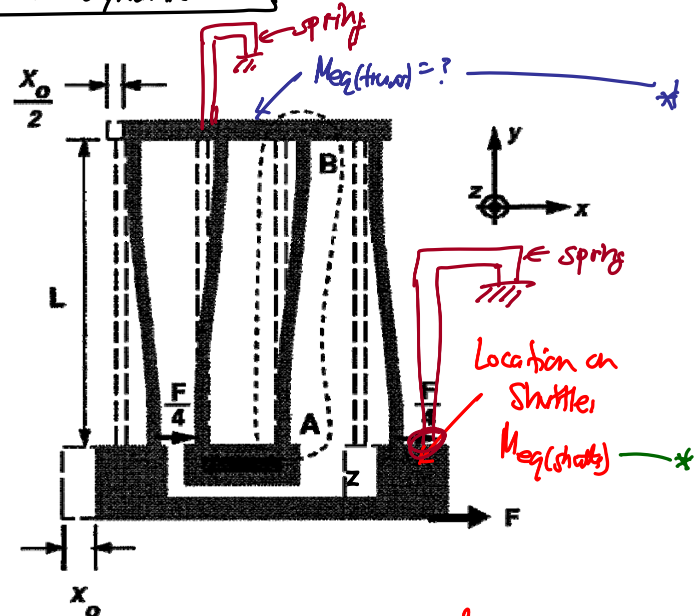
$$\omega_o = \left[\frac{k_x}{M_{eq}}\right]^{1/2}$$ (k_c)

where $M_{eq} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b$

(Resonance Freq. of a Folded-Beam Suspended Shuttle)

- Go through Module 12 slides 12-31

Equivalent Dynamic Mass



Equivalent Mass:

Equiv. Mass: $$M_{eq,x} = \frac{K_{max}}{\frac{1}{2}v_x^2} = \frac{\frac{1}{2}\rho A\int_0^\ell v^2(x)\,dx}{\frac{1}{2}v_x^2}$$

(v_x: velocity @ location x)

$$*\; M_{eq(shuttle)} = \frac{K_{max}}{\frac{1}{2}V_{shuttle}^2} = \frac{\omega_0^2 x_0^2 \left(\frac{1}{2}\right)\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]}{\frac{1}{2}\omega_0^2 x_0^2}$$

$$M_{eq(shuttle)} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b$$

mass mod. factors

static masses: $\propto$ (Volume)

$$*\; M_{eq(truss)} = \frac{\omega_0^2 x_0^2 \left(\frac{1}{2}\right)\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]}{\frac{1}{2}\left(\frac{1}{4}\right)\omega_0^2 x_0^2}$$

$$M_{eq(truss)} = 4\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]$$

Equiv. Dynamic Mass

Equiv. Dynamic Stiffness

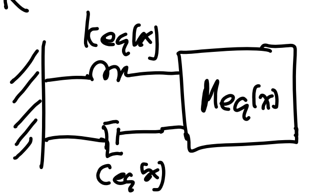
$$\omega_0 = \sqrt{\frac{k_{eq}(x)}{M_{eq}(x)}} \rightarrow K_{eq}(x) = \omega_0^2 M_{eq}(x)$$

→ large equiv. mass → large equiv. stiffness

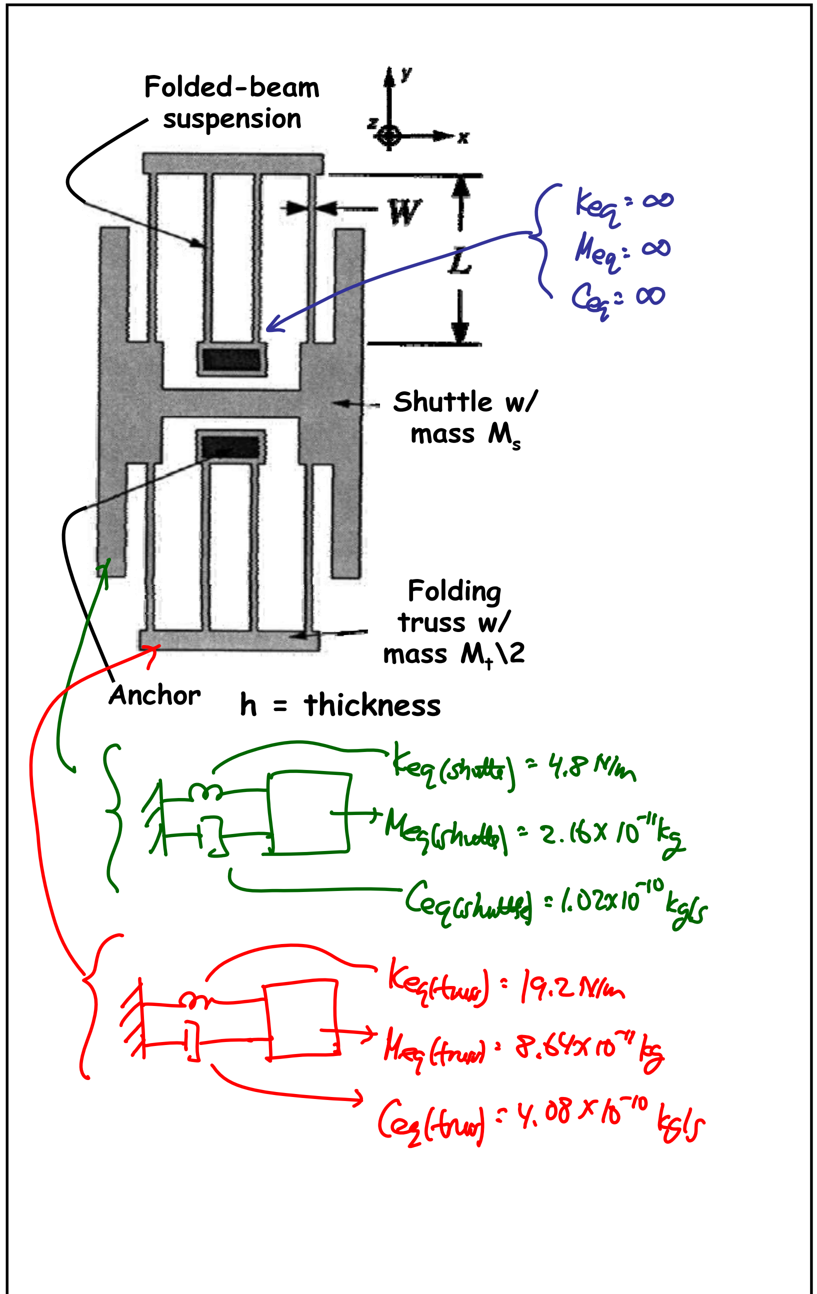
Equiv. Dynamic Damping

$$Q = \frac{\omega_0 M_{eq}(x)}{C_{eq}(x)} \quad (\sim L,\ \sim R) \rightarrow C_{eq}(x) = \frac{\omega_0 M_{eq}(x)}{Q} = \frac{\sqrt{K_{eq}(x) M_{eq}(x)}}{Q}$$

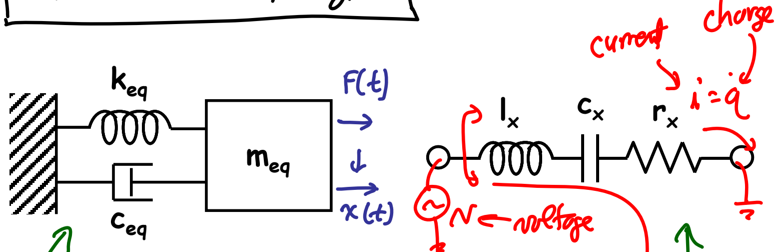
damping



specified @ a single location x



Electromechanical Analogies



$F(t) = F\cos(\omega t) \rightarrow x(t) = X\cos(\omega t)$

Equation of Motion:

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = F(t)$$

→ using phasor concepts:

$$F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + c_{eq}\dot{x}$$

Impedance looking into i:

$$\frac{v}{i} = j\omega l_x + \frac{1}{j\omega c_x} + r_x$$

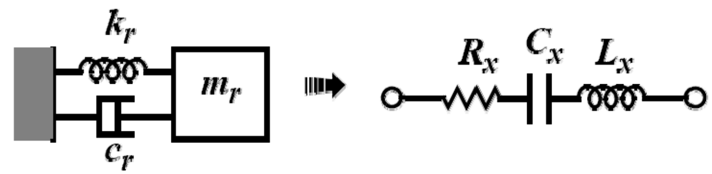
$$v = j\omega l_x i + \frac{(1/c_x)}{j\omega} i + r_x i$$

⇒ by analogy:

$F \rightarrow v$ $m_{eq} \rightarrow l_x$ $c_{eq} \rightarrow r_x$

$\dot{x} \rightarrow i$ $k_{eq} \rightarrow \frac{1}{c_x}$

[Parameter Relationships in the Current analogy]



- Mechanical-to-electrical correspondence in the current analogy:

Mechanical Variable	Electrical Variable
Damping, c	Resistance, R
Stiffness^{-1}, k^{-1}	Capacitance, C
Mass, m	Inductance, L
Force, f	Voltage, V
Velocity, v	Current, I