Lecture 17: Pull In Voltage

- Announcements:
- HW#5 online; due Wednesday, April 15
- Project slide #1 due Friday, April 10
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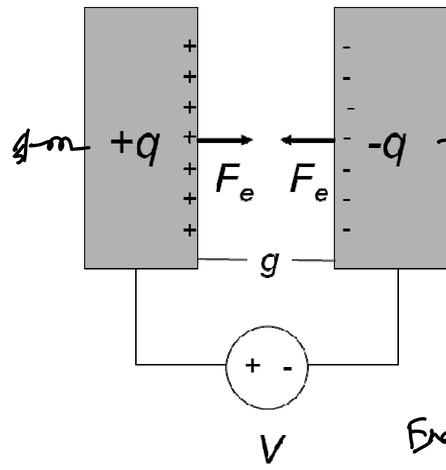
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
- ------------------------------------
- Last Time:
- Quick look at charge control behavior of two spring supported plates

over

Basic Physics of Electrostatic Actuation



Note: Assume the plates are supported elastically

Goal: Determine gap spacing g as a function of input variables.

Energy defines the needed relationships.

1st: Determine the energy of the system.

2nd: Ask, what can be done to Δ the energy of the system?

① Change the charge q.

② Change the separation g.

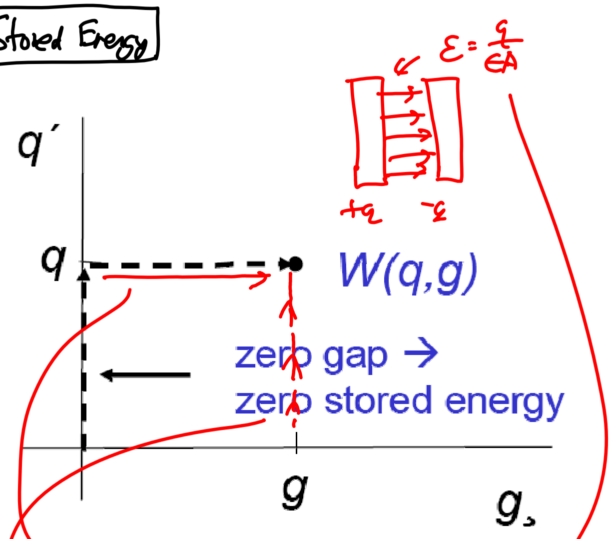
$$\Delta W(q,g) = V\Delta q + F_e \Delta g$$

$$dW = Vdq + F_e\, dg \leftarrow$$

hold $q = \text{const.} \rightarrow Vdq \rightarrow 0$

$$dW = F_e\, dg \rightarrow F_e = \left.\frac{dW}{dg}\right|_{q=\text{const.}}$$

Stored Energy



No change in charge: $dq = 0$

$$W = 0 + \int_0^g F_e\, dg'$$

$$F_e = \left(\frac{q}{2}\right)\mathcal{E} = \frac{1}{2}\frac{q^2}{\epsilon A} \quad \text{(independent of } g\text{)}$$

$$\therefore W = \int_0^g F_e\, dg' = F_e g'\Big|_0^g = F_e g$$

$$\boxed{W(g) = \frac{1}{2}\frac{q^2}{\epsilon A} g}$$

Energy change needed to charge a C to q at a fixed gap g.

$$dW = V dq + F_e \cancel{dg} \rightarrow 0$$

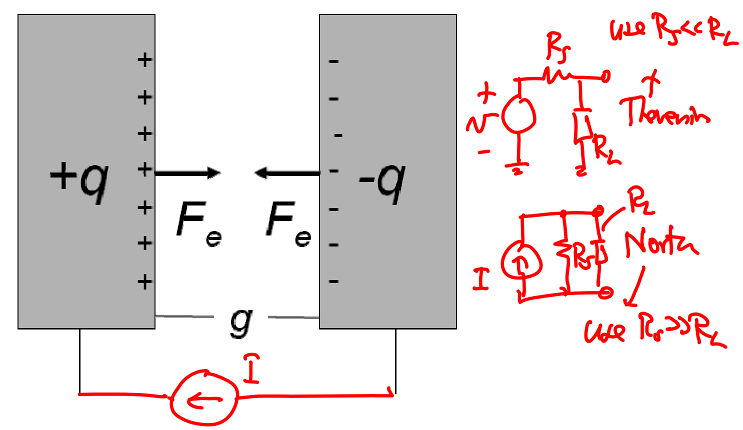
For a capacitor:

$$q = CV \rightarrow V = \frac{q}{C}$$

$$\therefore W(q) = \int_0^q V\, dq = \int_0^q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{q^2}{C}$$

$$\boxed{W(q) = \frac{1}{2}\frac{q^2}{\epsilon A} g}$$

Charge Control Case



From $dW = V dq + F_e dg$

⇒ Force is given by,

$$F_e = \frac{\partial W(q,g)}{\partial g}\Big|_{q=\text{const.}} = \frac{\partial}{\partial g}\left(\frac{1}{2}\frac{q^2}{\epsilon A} g\right)$$

$$\therefore \boxed{F_e = \frac{1}{2}\frac{q^2}{\epsilon A}} \quad \text{(indep. of gap spacing!)}$$

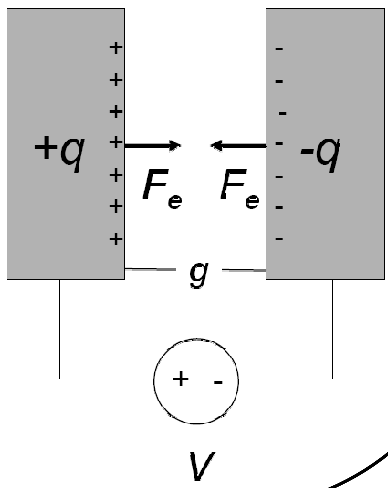
⇒ voltage is given by:

$$V = \frac{\partial \mathcal{W}(q,g)}{\partial q}\bigg|_{g=\text{const.}} = \frac{\partial}{\partial q}\left(\frac{1}{2}\frac{q^2}{\epsilon A}g\right)$$

$$= \frac{qg}{\epsilon A} \Rightarrow \boxed{V = \frac{q}{C}} \quad \checkmark$$

(consistent w/ what we already know)

Voltage Control



Would like to write $F_e = f(V)$.

We know this:

$d\mathcal{W} = Vdq + F_e dg$

$\mathcal{W} = \mathcal{W}(q,g)$

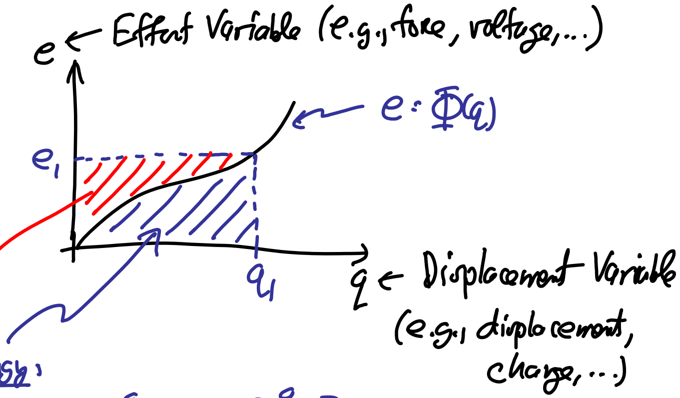
Energy

Need $\mathcal{W}'(V,g)$

replace charge q w/ V

Can get this using a Legendre transformation.

Energy & Co-Energy



Energy:

$$\mathcal{W}(q_1) = \int_0^{q_1} e\,dq = \int_0^{q_1} \Phi(q)\,dq$$

Co-Energy:

$$\mathcal{W}'(e_1) = \int_0^{e_1} q\,de = \int_0^{e_1} \Phi^{-1}(e)\,de$$

For a linear system, these will be equal

Can define co-energy also as:

$$\mathcal{W}'(e) = eq - \mathcal{W}(q) \quad \text{(from the plot)}$$

co-energy energy

$$\mathcal{W}'(V,g) = Vq - \mathcal{W}(q,g)$$

Differentially, this becomes

$$dW'(V,g) = (q\,dV + \cancel{V\,dq}) - dW(q,g)$$

$$\left[dW(q,g) = F_e\,dg + \cancel{V\,dq} \right]$$

$$\boxed{dW'(V,g) = q\,dV - F_e\,dg}$$ ← working co-energy expression

Find co-energy in terms of voltage, V:

$$W' = \int_0^V q(q,V')\,dV' = \int_0^V \left(\frac{\epsilon A}{g}\right) V'\,dV'$$

$$= \frac{1}{2}\left(\frac{\epsilon A}{g}\right)V^2 = \frac{1}{2}CV^2 \checkmark$$ (as expected)

Electrostatic (a Voltage-Controlled) Force:

$$F_e = -\left.\frac{\partial W'(V,g)}{\partial g}\right|_{V=\text{const.}} = \frac{1}{2}\left(\frac{\epsilon A}{g^2}\right)V^2$$

$$\rightarrow \boxed{F_e = \frac{1}{2}\frac{C}{g}V^2}$$

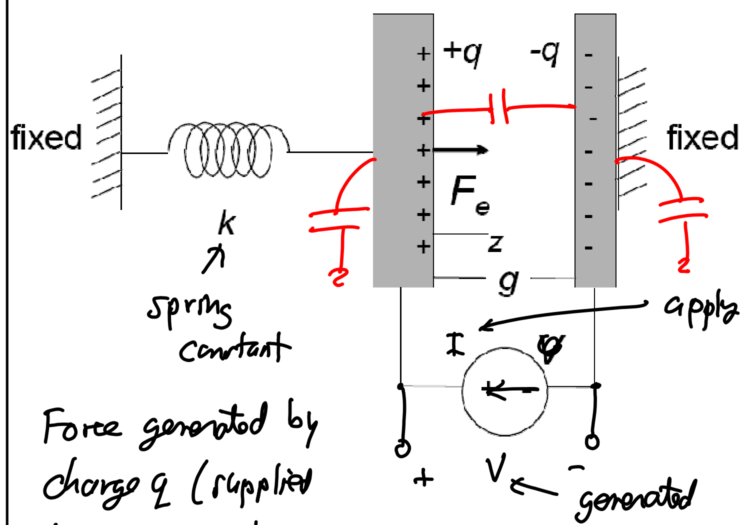
depends on gap!

Charge:

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_{g=\text{const.}} = \frac{\epsilon A}{g}V = CV \checkmark$$

(as expected)

Charge-Control of a Spring-Suspended Capacitor



Force generated by charge q (supplied by current I):

$$F_e = \left.\frac{\partial W(q,g)}{\partial g}\right|_{q=\text{const.}} = \frac{q^2}{2\epsilon A}$$

Restoring force of spring: $F_{spring} = kz = F_e$

equilibrium

The gap:

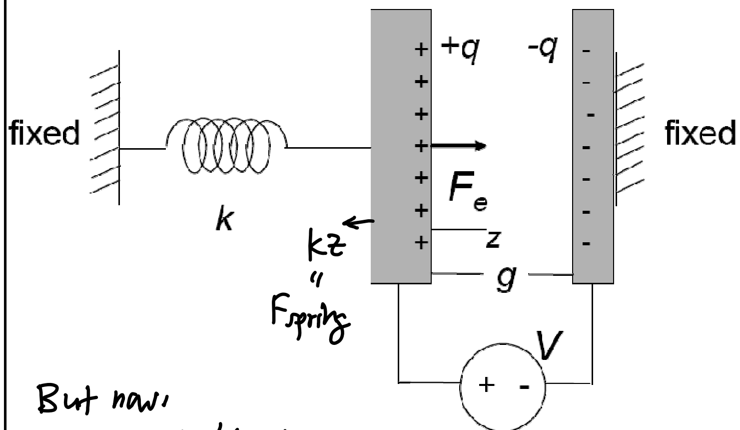
$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k} = g}$$

initial gap

q↑ can drive $g \to 0$ in a continuous fashion

$$V = \frac{q}{C} = \frac{q}{\epsilon A}g = \boxed{\frac{q}{\epsilon A}\left(g_0 - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k}\right) = V} \rightarrow V\downarrow \text{ as } g\downarrow$$

Voltage-Control of a Suspended C



But now:

$$F_e = \left.\frac{\partial W'(V,g)}{\partial g}\right|_{q=const} \rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \overbrace{\frac{1}{2}\frac{\epsilon A}{g^2}}^{F_e}\frac{V^2}{k} = g}$$

↑ initial gap spacing

g shows up on both sides!

If V↑ → g↓ → Fe↑ (+) Feedback!

If loop gain > 1, then this will go unstable!

↓

plate will collapse! (into the electrode)

Charge: (for a stable gap)

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_{g=const} = CV \checkmark \text{ (as expected)}$$

Stability Analysis

⇒ determine under what conditions voltage-control will cause collapse of the plates:

initial gap

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_0 - g)}_{F_{spring}}$$

Perturbation Analysis: What happens when I change g by a small increment dg?

↳ get an increment in the net attractive force F_{net}

$$\underset{(-)}{dF_{net}} = \frac{\partial F_{net}}{\partial g}dg = \left[-\frac{\epsilon A V^2}{g^3} + k\right]\underset{(-)}{dg}$$

If g↓ → dg = (-), then for stability need F_{net}↓ → dF_{net} = (-)

This needs to be (+)! (otherwise, collapse)

Thus: $\boxed{k > \frac{\epsilon A V^2}{g^3}}$ (for a stable uncollapsed system)

Pull-in Voltage & Pull-in Gap

$V_{PI} \triangleq$ voltage @ which plates collapse

$g_{PI} \triangleq$ gap @ " " "

The plates go unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \quad (1)$$

$$F_{net} = 0 = \frac{\epsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_0 - g_{PI}) \quad (2)$$

Substitute (1) into (2):

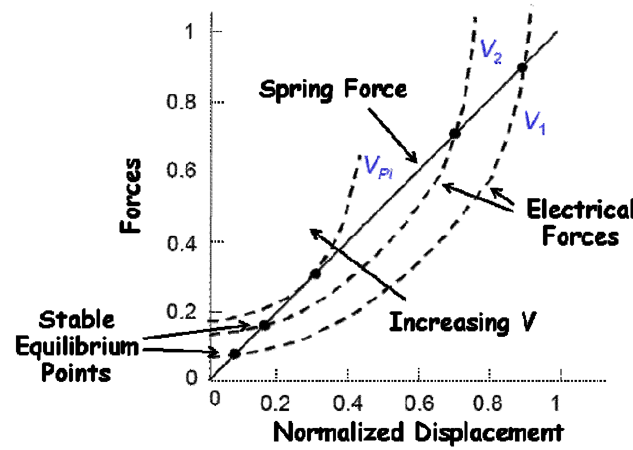
$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2 \cancel{g_{PI}^2}} - \frac{\cancel{\epsilon A V_{PI}^2}}{\cancel{g_{PI}^3}}(g_0 - g_{PI})$$

$$\frac{g_0 - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_0 = \frac{3}{2} g_{PI}$$

$$\therefore \boxed{g_{PI} = \frac{2}{3} g_0}$$

when the gap is driven by a voltage to (2/3) the initial gap → collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \rightarrow V_{PI} = \sqrt{\frac{8}{27} \frac{k g_0^3}{\epsilon A}}$$



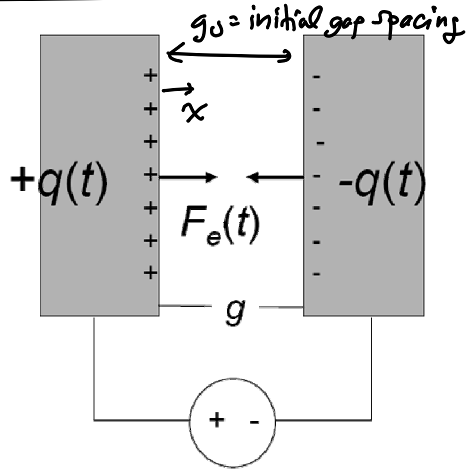
<u>Advantages of Electrostatic Actuators</u>:

- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

Disadvantages of Electrostatic Actuators:

- Nonlinear voltage-to-force transfer function
- ~~Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale~~

$\boxed{\text{Linearizing the Voltage-to-Force Transfer Function}}$



$$v(t) = \underbrace{V_P}_{\text{Bias (DC)}} + \underbrace{v_i(t)}_{\text{Signal (AC)}}$$

$$F_e(t) = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}C[v(t)]^2\right]$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}[v(t)]^2 = \frac{1}{2}\frac{\partial C}{\partial x}(V_P + v_i(t))^2$$

$$= \frac{1}{2}\left[V_P^2 + 2V_P v_i(t) + \cancelto{0}{[v_i(t)]^2}\right]\frac{\partial C}{\partial x}$$

$$[V_P \gg v_i(t)] \Rightarrow F_e(t) = \underbrace{\frac{1}{2}V_P^2\frac{\partial C}{\partial x}}_{\text{DC Offset}} + \underbrace{V_P\frac{\partial C}{\partial x}v_i(t)}_{\text{AC drive signal}}$$

What's $\frac{\partial C}{\partial x}$?

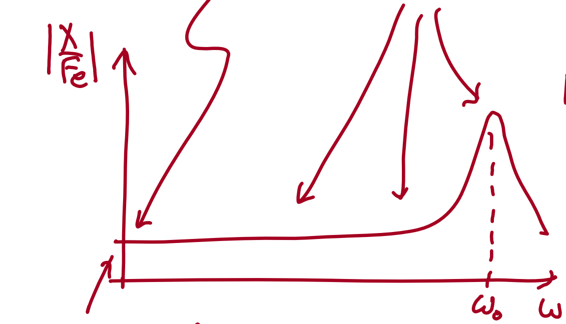
$$C_0 = \frac{\epsilon A}{g_0} \rightarrow C(x) = \frac{\epsilon A}{g_0 - x} = C_0\left(1 - \frac{x}{g_0}\right)^{-1}$$

$$[x \ll g_0] \Rightarrow \approx C_0\left(1 + \frac{x}{g_0}\right) \doteq C(x)$$

$$\therefore \frac{\partial C}{\partial x} = \frac{C_0}{g_0} = \frac{\epsilon A}{g_0^2}$$

$$\Rightarrow F_e(t) = \underbrace{\frac{1}{2}\frac{C_0}{g_0}V_P^2}_{\text{DC Offset}} + \underbrace{V_P\frac{C_0}{g_0}v_i(t)}_{\text{Signal Component}}$$

this is a linear dependence on $v_i(t)$ → holds for small amplitudes



very small response → but must still worry about pull in V_{PI}