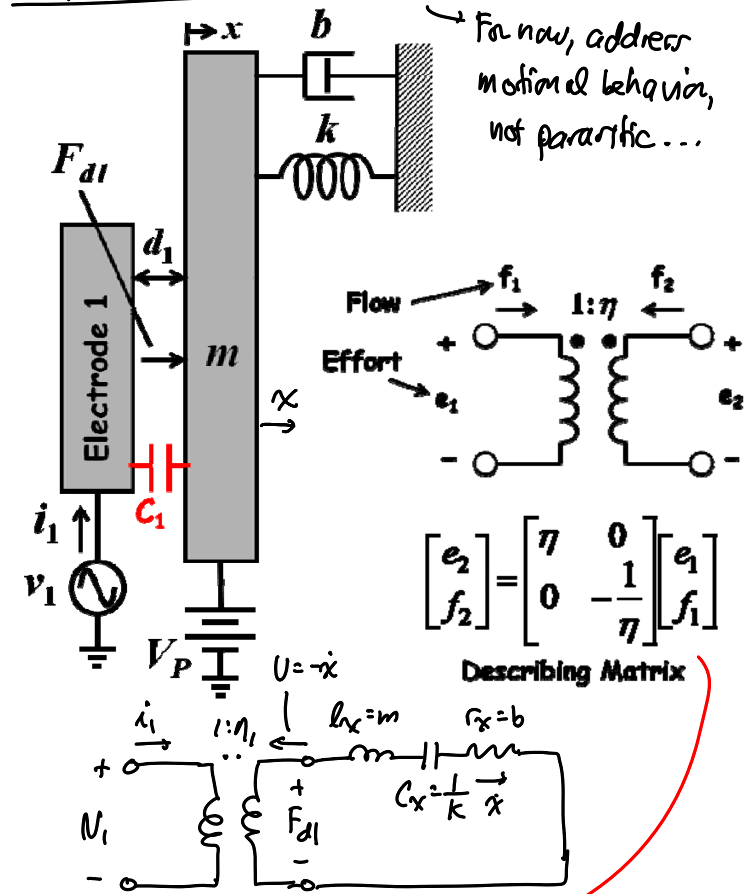
## Lecture 19: Equivalent Circuits II

- Announcements:
- HW#5 online; due Wednesday, April 15
- Module 13 on Equivalent Circuits II online
- Project slide #1 due Friday, April 10
- I will be on travel next week; will either record lectures or do make-up lectures next Friday
- Will announce via Piazza
- -----------------------------------
- Reading: Senturia, Chpt. 6, Chpt. 14
- Lecture Topics:
  - Input Modeling
    - Force-to-Velocity Equiv. Ckt.
    - Input Equivalent Ckt.
  - Current Modeling
    - Output Current Into Ground
    - Input Current
    - Complete Electrical-Port Equiv. Ckt.
  - Impedance & Transfer Functions
- -----------------------------------
- Last Time:
- Finished comb-drive
- Project Notes:

| Parameter | Today (SOA) | After Scaling | Factor |
|---|---|---|---|
| Speed | 60s | 1ms | |

## Input Electrical Equiv. Ckt.

For now, address motional behavior, not parasitic...




$$\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ 0 & -\frac{1}{\eta} \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \end{bmatrix}$$

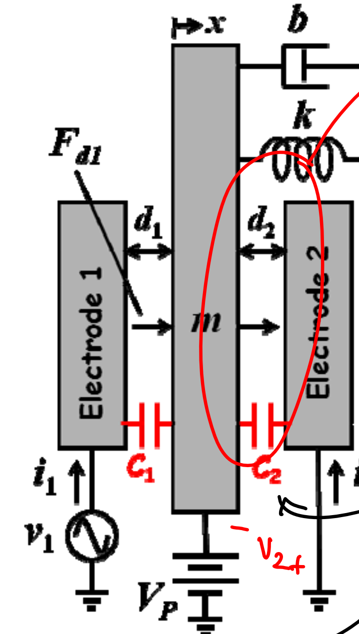
Describing Matrix

$$\underbrace{F_{d1}}_{e_2} = -\underbrace{V_P \frac{\partial C_1}{\partial x}}_{\eta_1} \underbrace{v_1}_{e_1}$$

$e_2 = \eta e_1$

$$\boxed{\eta_1 = V_P \frac{\partial C_1}{\partial x} = V_P \frac{\epsilon_0 A_1}{d_1^2}}$$

## Output Current Into Ground

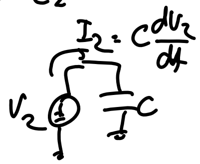



Want to model this!

$[q = CV]$

$i = \frac{dq}{dt} = C\frac{dV}{dt} + V\frac{dC}{dt}$

$C_2 = f(t)$




$i_2 = C_2(x,t)\cancel{\frac{dV_2(t)}{dt}}^{\,0} + V_2(t)\frac{dC_2(x,t)}{dt}$

$[V_2(t) = -V_P] \Rightarrow i_2 = -V_P\frac{dC_2}{dt} = -V_P\frac{\partial C_2}{\partial x}\frac{\partial x}{\partial t}$

In phasor form: $I_2(j\omega) = -V_P\frac{\partial C_2}{\partial x}(j\omega X)$

$$\boxed{I_2(j\omega) = -j\omega V_P\frac{\partial C_2}{\partial x}X}$$

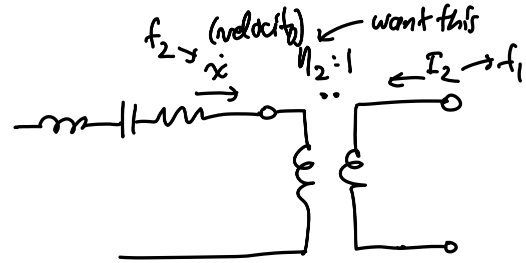
motional current

---

$I_2(j\omega) = -j\omega V_P\frac{\partial C_2}{\partial x}X = -V_P\frac{\partial C_2}{\partial x}\dot{x}$

velocity

90° phase lag (+) (+) → $I_2 = (-)$ when $x = (+)$ ✓



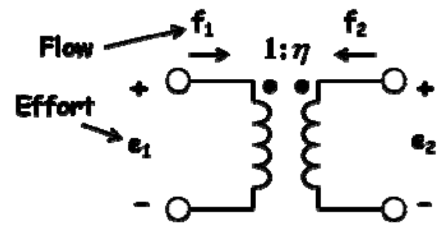

$f_2 = -\frac{1}{\eta_2}f_1$

$f_1 = -\eta_2 f_2$

$[f_1 = F_2,\ f_2 = \dot{x}]$

$I_2 = -\eta_2\dot{x}$

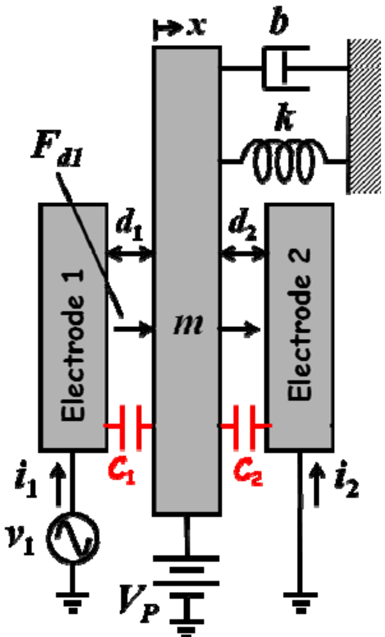
$\therefore \boxed{\eta_2 = V_P\frac{\partial C_2}{\partial x}}$




$$\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ 0 & -\frac{1}{\eta} \end{bmatrix}\begin{bmatrix} e_1 \\ f_1 \end{bmatrix}$$

**Describing Matrix**

## Input Current Expression (more complete)




Get $I_1(j\omega)$:

$$i_1(t) = C_1(x,t)\frac{dV_1(t)}{dt} + V_1(t)\frac{dC(x,t)}{dt}$$

$$[V_1(t) = \underset{f(t)}{v_1} - V_P] \Rightarrow i_1 = C_1\frac{dv_1}{dt} + [v_1 - V_P]\frac{\partial C_1}{\partial x}\frac{\partial x}{\partial t}$$

$$\therefore I_1(j\omega) = \underbrace{j\omega C_1 V_1}_{\text{Feedthrough Current}} + \cancel{j\omega V_1\frac{\partial C_1}{\partial x}X} - \underbrace{j\omega V_P\frac{\partial C_1}{\partial x}}_{\text{Motional Current (due to mass motion)}}X$$

what's X?

@ DC: $x = \frac{F_{d1}}{k} = -\frac{1}{k}V_P\left(\frac{\partial C_1}{\partial x}\right)v_1$

@ resonance: $x = \frac{QF_{d1}}{jk} = -\frac{Q}{jk}V_P\frac{\partial C_1}{\partial x}v_1 = X$

Thus: (@ resonance) $\rightarrow \omega_o$

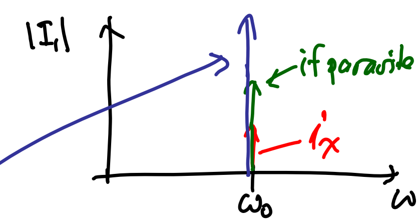
$$I_1(j\omega) = j\omega_o C_1 V_1 + \cancel{j}\omega_o\left(V_P\frac{\partial C_1}{\partial x}\right)^2\frac{Q}{\cancel{j}k}$$

$$= \underbrace{j\omega_o C_1 V_1}_{\text{90° phase-shifted from } V_1} + \underbrace{\omega_o\frac{Q}{k}\eta_{e1}^2 V_1}_{\text{In phase w } V_1}$$

parasitic, $i_p$

motional current $i_x$

→ on a spectrum analyzer:




if parasitic too big → can mask the $i_x$!

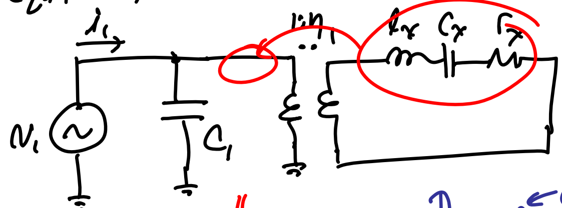
but it's 90° phase-shifted

can lock-in to separate $i_x$ from $i_p$

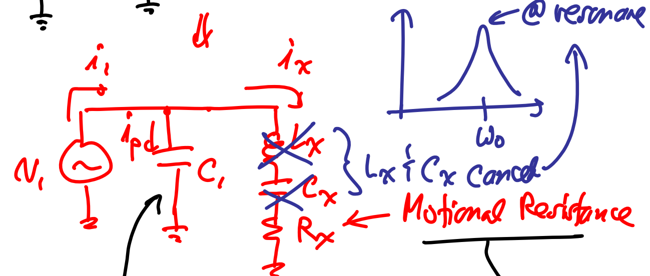
better solution: $i_x \gg i_p$

get $\eta_{e1}\uparrow$

The equiv. ckt. becomes:




&




$$I_1(j\omega) = j\omega_0 C_1 v_1 + \underbrace{\omega_0 \frac{Q}{k}\eta_{e1}^2 v_1}_{\text{resistive (no } j\omega)}$$

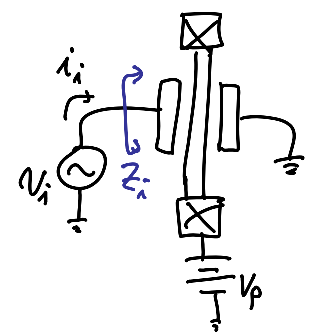
Motional Resistance:

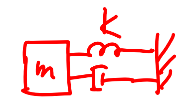
$$R_{x1} = \frac{v_1}{I_1}\Big|_{@\omega_0} = \boxed{\frac{k}{\omega_0 Q \eta_{e1}^2} = \frac{m\omega_0}{Q\eta_{e1}^2} = \frac{b}{\eta_{e1}^2} = R_{x1}}$$

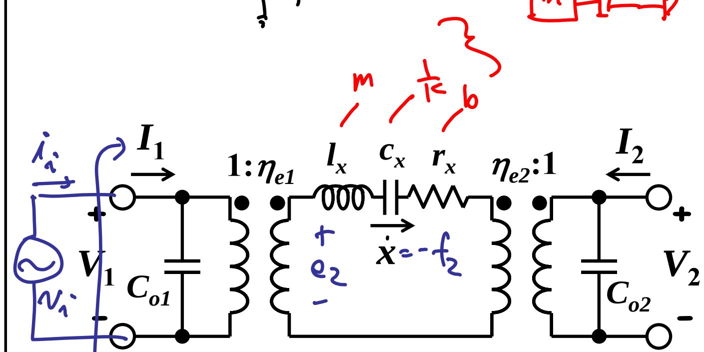
↳ equiv. ckt. gets this right! ✓

- Look at Module 13, slide 16

## Input Impedance Into Port 1




$$Z_i = \frac{v_i}{i_i} = \frac{e_1}{f_1}$$

$$\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ 0 & -\frac{1}{\eta} \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \end{bmatrix} \Rightarrow \begin{aligned} e_2 &= \eta e_1 \rightarrow e_1 = \frac{e_2}{\eta} \\ f_2 &= -\frac{1}{\eta} f_1 \rightarrow f_1 = -\eta f_2 \end{aligned}$$

$$\frac{e_1}{f_1} = \frac{e_2}{\eta}\left(\frac{1}{-\eta f_2}\right) = -\frac{1}{\eta^2}\frac{e_2}{f_2} \rightarrow \frac{v_i}{i_i} = Z_i = -\frac{1}{\eta_{e1}^2}\frac{F_{d2}}{(-\dot{x}_2)}$$

$$= \frac{1}{\eta_{e1}^2} Z_x$$

$$\therefore Z_i = \frac{1}{\eta_{e1}^2}\left(j\omega l_x + \frac{1}{j\omega c_x} + r_x\right)$$

$$= \underbrace{j\omega\left(\frac{l_x}{\eta_{e1}^2}\right)}_{L_{x1}} + \underbrace{\frac{1}{j\omega(\eta_{e1}^2 c_x)}}_{C_{x1}} + \underbrace{\frac{r_x}{\eta_{e1}^2}}_{R_{x1}}$$

to model noise!

$\overline{v_n^2} = 4kTR_x$ (Brownian Motion)


Purely Electrical Equiv. Ckt.