EE C247B – ME C218
Introduction to MEMS Design
Fall 2015

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Module 17: Noise & MDS

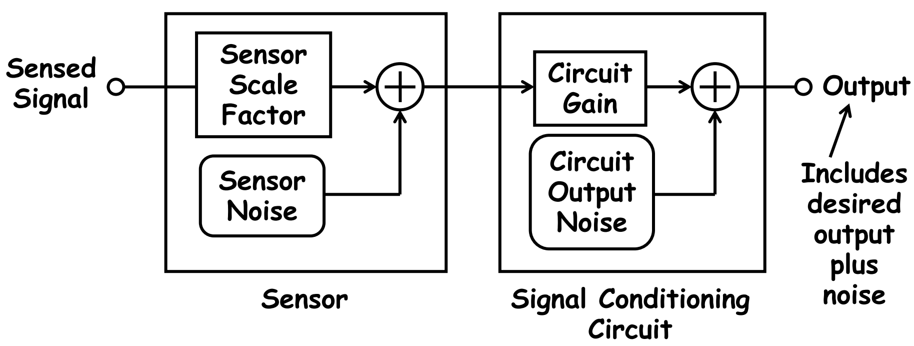
Lecture Outline

- Reading: Senturia Chpt. 16
- Lecture Topics:
  - Minimum Detectable Signal
  - Noise
    - Circuit Noise Calculations
    - Noise Sources
    - Equivalent Input-Referred Noise
  - Gyro MDS
    - Equivalent Noise Circuit
    - Example ARW Determination

Determining Sensor Resolution

Minimum Detectable Signal (MDS)

- Minimum Detectable Signal (MDS): Input signal level when the signal-to-noise ratio (SNR) is equal to unity

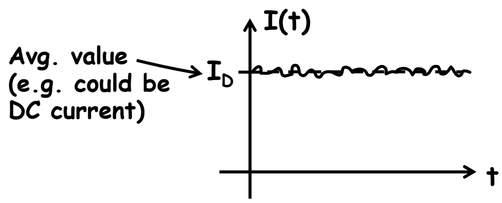


- The sensor scale factor is governed by the sensor type
- The effect of noise is best determined via analysis of the equivalent circuit for the system

Noise

Noise

- Noise: Random fluctuation of a given parameter I(t)
- In addition, a noise waveform has a zero average value



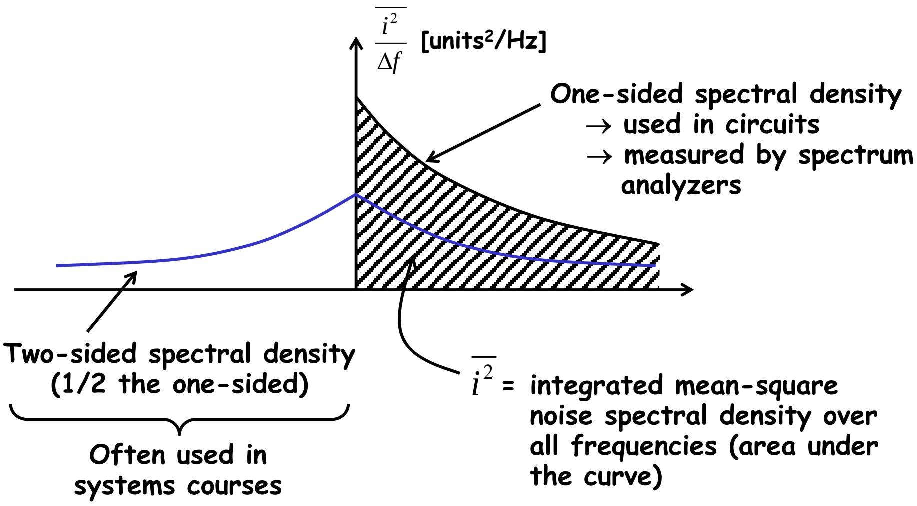
- We can't handle noise at instantaneous times
- But we can handle some of the averaged effects of random fluctuations by giving noise a power spectral density representation
- Thus, represent noise by its mean-square value:

Let $i(t) = I(t) - I_D$

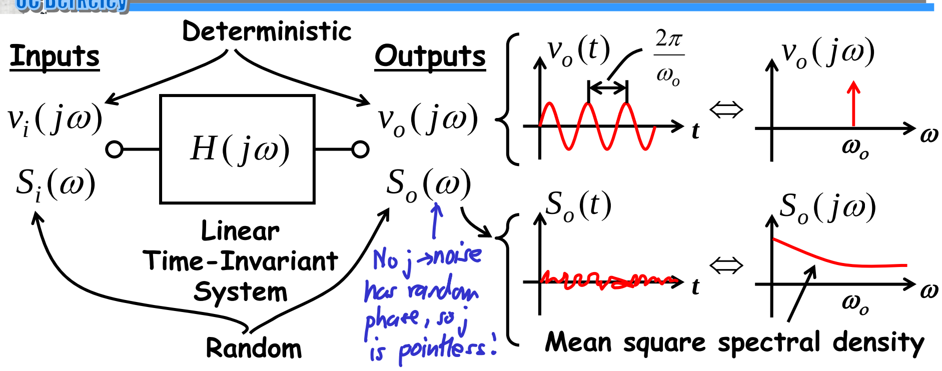
Then $$\overline{i^2} = \overline{\left(I - I_D\right)^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left|I - I_D\right|^2 dt$$

Noise Spectral Density

- We can plot the spectral density of this mean-square value:



Circuit Noise Calculations



- **Deterministic:** $v_o(j\omega) = H(j\omega)v_i(j\omega)$

- **Random:** $S_o(\omega) = \left[H(j\omega)H^*(j\omega)\right]S_i(\omega) = \left|H(j\omega)\right|^2 S_i(\omega)$

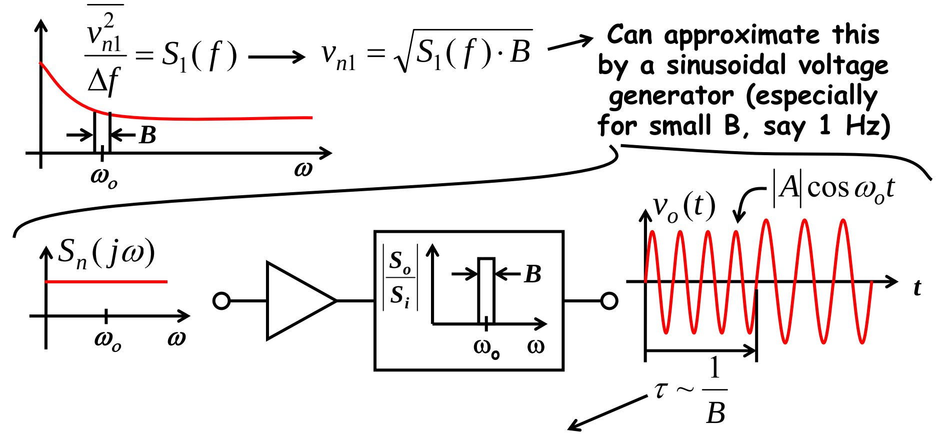
$$\underbrace{\sqrt{S_o(\omega)} = \left|H(j\omega)\right|\sqrt{S_i(\omega)}}_{\text{Root mean square amplitudes}} \longrightarrow$$ How is it we can do this?

Handling Noise Deterministically

- Can do this for noise in a tiny bandwidth (e.g., 1 Hz)

$$\frac{\overline{v_{n1}^2}}{\Delta f} = S_1(f) \longrightarrow v_{n1} = \sqrt{S_1(f)\cdot B}$$

Can approximate this by a sinusoidal voltage generator (especially for small B, say 1 Hz)

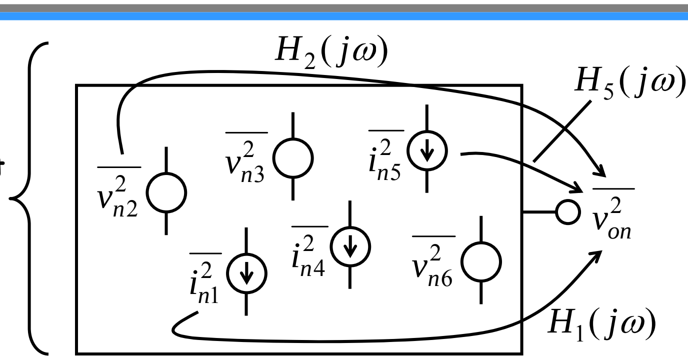


[This is actually the principle by which oscillators work → oscillators are just noise going through a tiny bandwidth filter]

Why? Neither the amplitude nor the phase of a signal can change appreciably within a time period 1/B.

Systematic Noise Calculation Procedure

General Circuit With Several Noise Sources



- Assume noise sources are uncorrelated
  1. For $\overline{i_{n1}^2}$, replace w/ a deterministic source of value

$$i_{n1} = \sqrt{\frac{\overline{i_{n1}^2}}{\Delta f}\cdot(1\,\text{Hz})}$$

Systematic Noise Calculation Procedure

2. Calculate $v_{on1}(\omega) = i_{n1}(\omega)H(j\omega)$ (treating it like a deterministic signal)
3. Determine $\overline{v_{on1}^2} = \overline{i_{n1}^2} \cdot |H(j\omega)|^2$
4. Repeat for each noise source: $\overline{i_{n1}^2}$, $\overline{v_{n2}^2}$, $\overline{v_{n3}^2}$
5. Add noise <u>power</u> (mean square values)

$$\overline{v_{onTOT}^2} = \overline{v_{on1}^2} + \overline{v_{on2}^2} + \overline{v_{on3}^2} + \overline{v_{on4}^2} + \cdots$$

$$v_{onTOT} = \sqrt{\overline{v_{on1}^2} + \overline{v_{on2}^2} + \overline{v_{on3}^2} + \overline{v_{on4}^2} + \cdots}$$

Total rms value

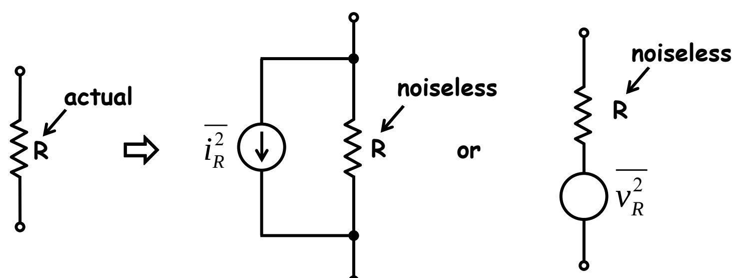
Noise Sources

Thermal Noise

- Thermal Noise in Electronics: (Johnson noise, Nyquist noise)
  - Produced as a result of the thermally excited random motion of free e⁻'s in a conducting medium
  - Path of e⁻'s randomly oriented due to collisions
- Thermal Noise in Mechanics: (Brownian motion noise)
  - Thermal noise is associated with all dissipative processes that couple to the thermal domain
  - Any damping generates thermal noise, including gas damping, internal losses, etc.
- Properties:
  - Thermal noise is white (i.e., constant w/ frequency)
  - Proportional to temperature
  - Not associated with current
  - Present in any real physical resistor

Circuit Representation of Thermal Noise

- Thermal Noise can be shown to be represented by a series voltage generator $\overline{v_R^2}$ or a shunt current generator $\overline{i_R^2}$



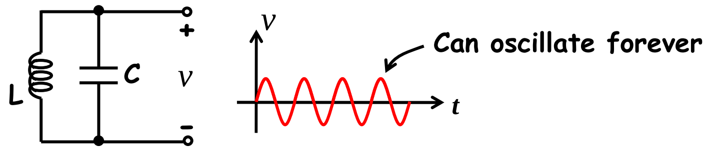
Note: These are one-sided mean-square spectral densities! To make them 2-sided, must divide by 2.

$$\boxed{\frac{\overline{i_R^2}}{\Delta f} = \frac{4kT}{R}} \qquad \boxed{\frac{\overline{v_R^2}}{\Delta f} = 4kTR}$$

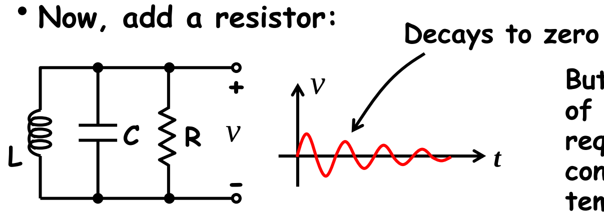
where $4kT = 1.66x10^{-20} V \cdot C$
and where these are spectral densities.

Noise in Capacitors and Inductors?

- Resistors generate thermal noise
- Capacitors and inductors are noiseless → why?



- Now, add a resistor:



But this violates the laws of thermodynamics, which require that things be in constant motion at finite temperature

Need to add a forcing function, like a noise voltage $\overline{v_R^2}$ to keep the motion going → and this noise source is associated with R

Why 4kTR?

- Why is $\overline{v_R^2} = 4kTR\Delta f$ (a heuristic argument)
- The Equipartition Theorem of Statistical Thermodynamics says that there is a mean energy (1/2)kT associated w/ each degree of freedom in a given system
- An electronic circuit possesses two degrees of freedom:
  - Current, i, and voltage, v
  - Thus, we can write:

$$\frac{1}{2}L\overline{i^2} = \frac{1}{2}k_BT \quad , \quad \overbrace{\frac{1}{2}C\overline{v^2}}^{\text{Energy}} = \frac{1}{2}k_BT$$

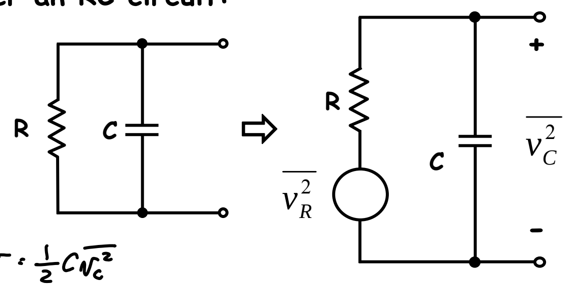
- Similar expressions can be written for mechanical systems
  - For example: for displacement, x

Spring constant

$$\frac{1}{2}k\overline{x^2} = \frac{1}{2}k_BT$$

Why 4kTR? (cont)

- Why is $\overline{v_R^2} = 4kTR\Delta f$? (a heuristic argument)
- Consider an RC circuit:



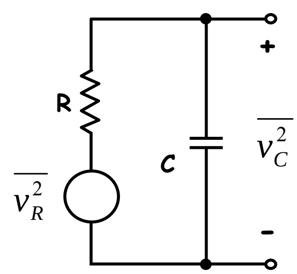
$E = \frac{1}{2}kT = \frac{1}{2}C\overline{v_c^2}$

$\therefore \overline{v_c^2} = \frac{kT}{C}$ ← integrated noise over all freqs.
(total mean square voltage integrated over all freqs.)

Question: What value of $\frac{\overline{v_R^2}}{\Delta f}$ (assuming white noise) gives us this?

Why 4kTR? (cont)

Question: What value of $\frac{\overline{v_R^2}}{\Delta f}$ (assuming white noise) gives us $\overline{v_c^2} = \frac{kT}{C}$?



$$\overline{v_c^2} = \int_0^\infty \left|\frac{1}{1+j\omega RC}\right|^2 \frac{\overline{v_R^2}}{\Delta f}\, df$$

[noise is white] [$\omega_b = \frac{1}{RC}$]

$$= \frac{1}{2\pi}\frac{\overline{v_R^2}}{\Delta f}\int_0^\infty \frac{\omega_b^2}{\omega_b^2+\omega^2}\, d\omega$$

$$\left[\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

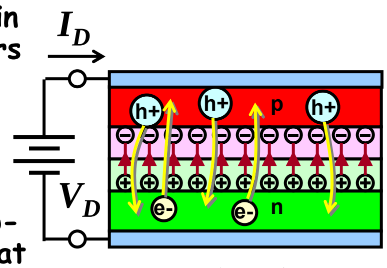
$$= \frac{1}{2\pi}\frac{\overline{v_R^2}}{\Delta f}\frac{\omega_b^2}{\omega_b}\tan^{-1}\left(\frac{\omega}{\omega_b}\right)\Bigg|_0^\infty = \frac{1}{2\cancel{\pi}}\frac{\overline{v_R^2}}{\Delta f}\left(\frac{\cancel{\pi}}{2}\omega_b - 0\right)$$

$$= \frac{1}{4}\omega_b\frac{\overline{v_R^2}}{\Delta f} = \frac{kT}{C} \longrightarrow \frac{\overline{v_R^2}}{\Delta f} = 4kT\left(\frac{\omega_b}{C}\right) \Rightarrow \boxed{\frac{\overline{v_R^2}}{\Delta f} = 4kTR} \checkmark$$

$\uparrow$ R

Shot Noise

- Associated with direct current flow in diodes and bipolar junction transistors
- Arises from the random nature by which e^-'s and h^+'s surmount the potential barrier at a pn junction
- The DC current in a forward-biased diode is composed of h^+'s from the p-region and e^-'s from the n-region that have sufficient energy to overcome the potential barrier at the junction → noise process should be proportional to DC current
- Attributes:
  - Related to DC current over a barrier
  - Independent of temperature
  - White (i.e., const. w/ frequency)
  - Noise power ~ I_D & bandwidth



pn-junction

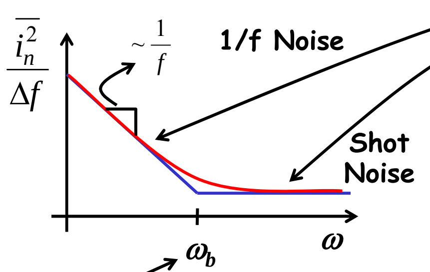
$$\boxed{\frac{\overline{i_n^2}}{\Delta f} = 2qI_D}$$

Charge on an e^- (=1.6x10^{-19}C)

DC Current

Flicker (1/f) Noise

- In general, associated w/ random trapping & release of carriers from "slow" states
- Time constant associated with this process gives rise to a noise signal w/ energy concentrated at low frequencies
- Often, get a mean-square noise spectral density that looks like this:



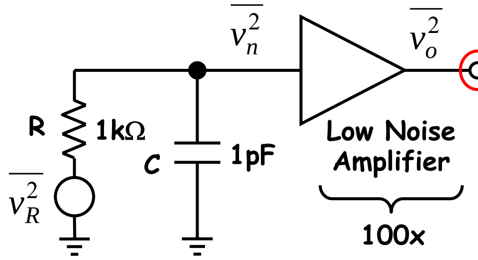
1/f Noise Corner Frequency

$$\boxed{\frac{\overline{i_n^2}}{\Delta f} = 2qI_D + K\left(\frac{I_D^a}{f^b}\right)}$$

I_D = DC current
K = const. for a particular device
a = 0.5 → 2
b ~ 1

Example: Typical Noise Numbers

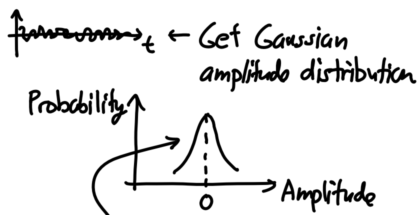
- Hookup the circuit below and make some measurements



Measure w/ AC voltmeter

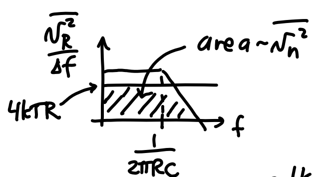
Measure w/ spectrum analyzer

← Get Gaussian amplitude distribution



68% within $\pm\sigma$

99.7% within $\pm 3\sigma$

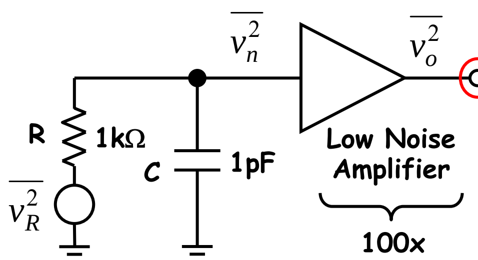


$\sqrt{(1.66\times10^{-20})(1k)}$ → $1k\Omega$: $4nV/\sqrt{Hz}$ (for every 1K of R)

$1pF$: $\sqrt{\frac{kT}{C}} = 64\mu V$ rms

Example: Typical Noise Numbers

- Hookup the circuit below and make some measurements



Measure w/ AC voltmeter

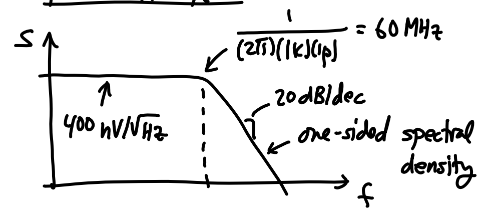
Measure w/ spectrum analyzer

AC Voltmeter

$\sqrt{\overline{v_o^2}} = (100)(64\mu V\ rms)$

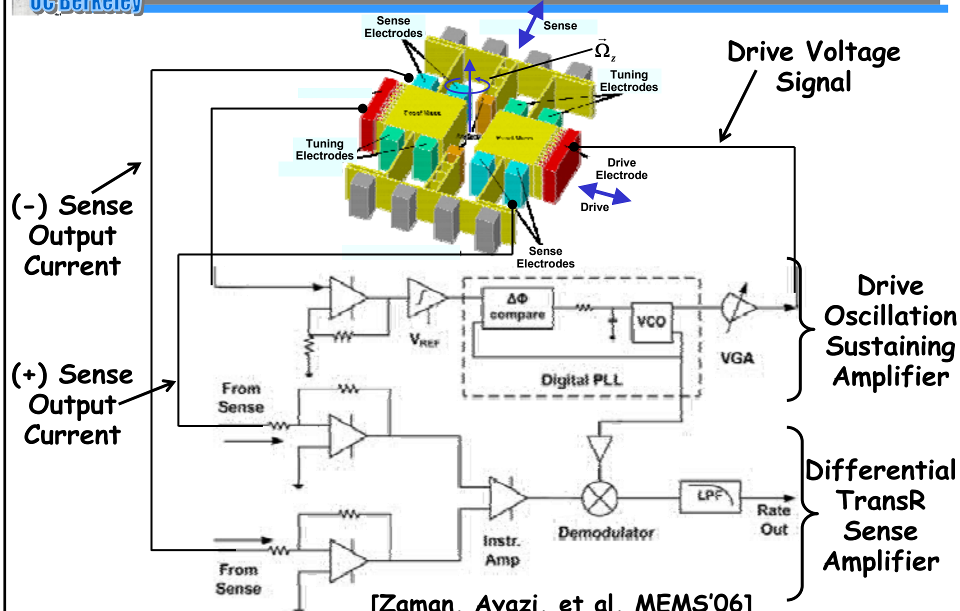
$= 6.4 mV\ rms$

Spectrum Analyzer



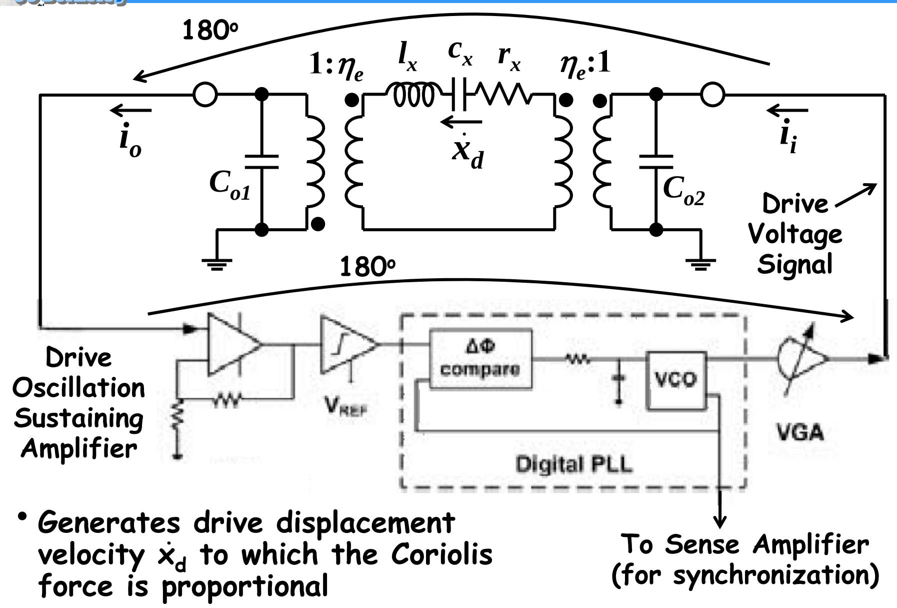
Back to Determining Sensor Resolution

MEMS-Based Tuning Fork Gyroscope



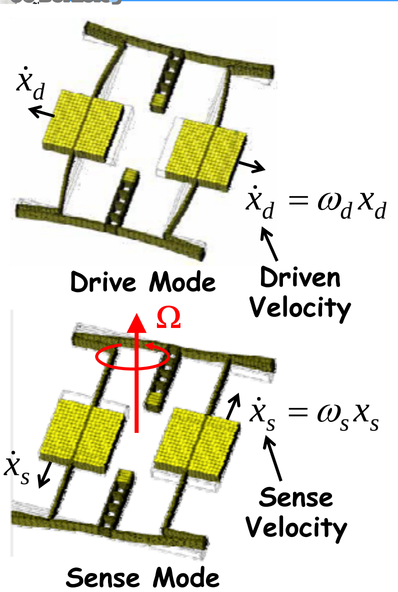
[Zaman, Ayazi, et al, MEMS'06]

Drive Axis Equivalent Circuit

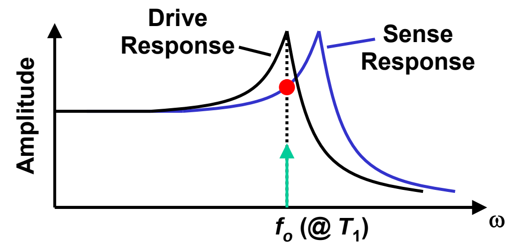


- Generates drive displacement velocity $\dot{x}_d$ to which the Coriolis force is proportional

Drive-to-Sense Transfer Function



Drive/Sense Response Spectra:



Rotation-Induced Coriolis Force:

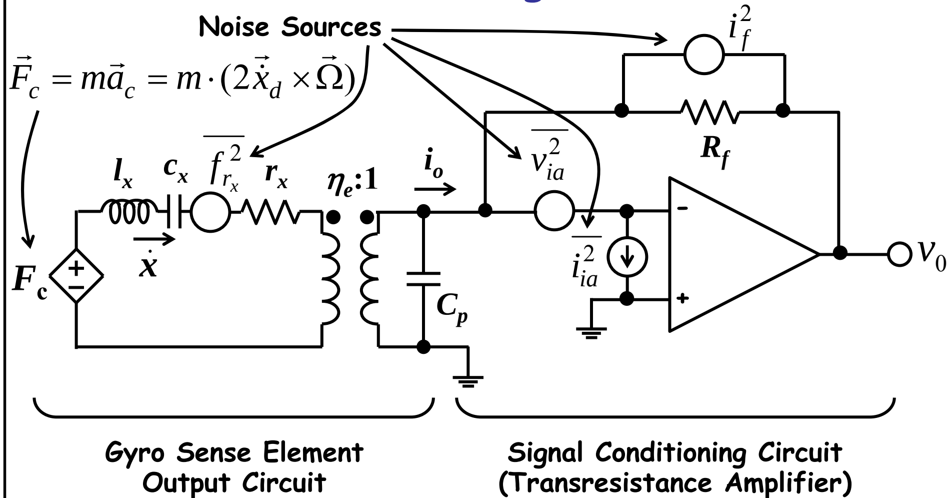
$\vec{a}_c = 2\vec{\dot{x}}_d \times \vec{\Omega} = 2\omega_d \vec{x}_d \times \vec{\Omega}$

$\Rightarrow$ Acts in the sense mode direction

$a_s = 2\omega_d x_d \Omega \sin 90°$

$\boxed{a_s = 2\omega_d x_d \Omega}$

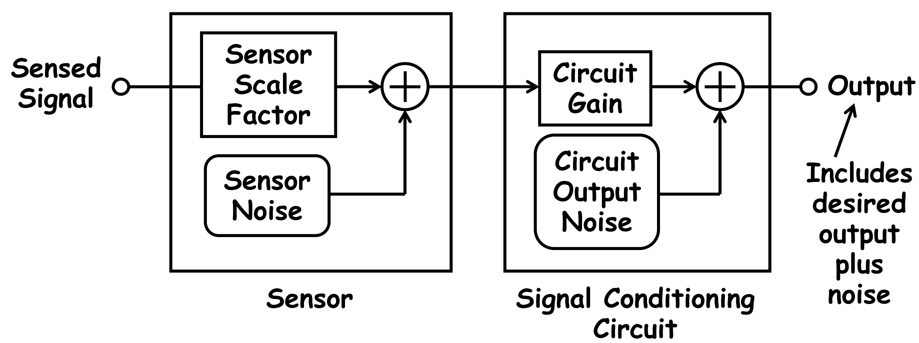
Gyro Readout Equivalent Circuit (for a single tine)



- Easiest to analyze if all noise sources are summed at a common node

Minimum Detectable Signal (MDS)

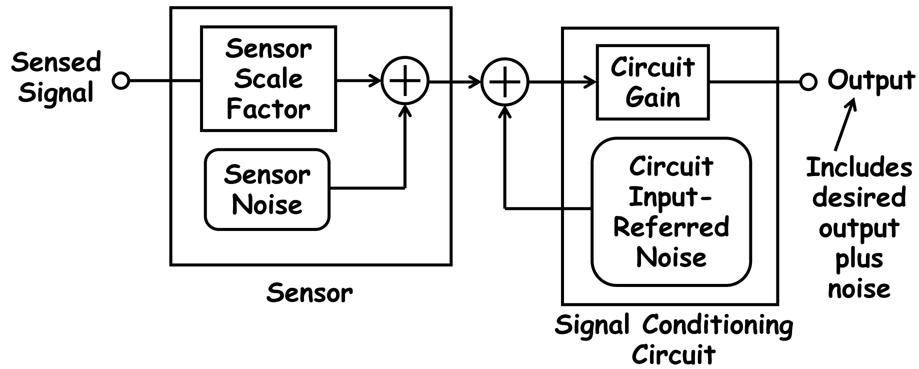
- Minimum Detectable Signal (MDS): Input signal level when the signal-to-noise ratio (SNR) is equal to unity



- The sensor scale factor is governed by the sensor type
- The effect of noise is best determined via analysis of the equivalent circuit for the system

Move Noise Sources to a Common Point

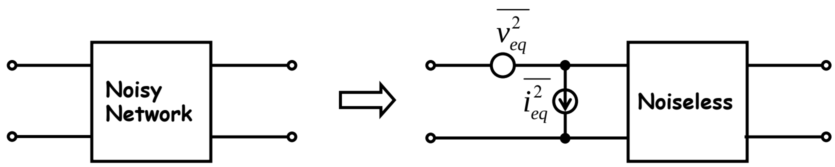
- Move noise sources so that all sum at the input to the amplifier circuit (i.e., at the output of the sense element)
- Then, can compare the output of the sensed signal directly to the noise at this node to get the MDS



Equivalent Input-Referred Voltage and Current Noise Sources

Equivalent Input v, i Noise Generators

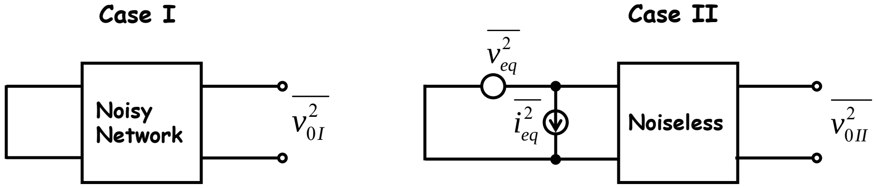
- Take a noisy 2-port network and represent it by a noiseless network with input *v* and *i* noise generators that generate the same total output noise



- **Remarks**:
  1. Works for linear time-invariant networks
  2. v_{eq} and i_{eq} are generally correlated (since they are derived from the same sources)
  3. In many practical circuits, one of v_{eq} and i_{eq} dominates, which removes the need to address correlation
  4. If correlation is important → easier to return to original network with internal noise sources

Calculation of $\overline{v_{eq}^2}$ and $\overline{i_{eq}^2}$

a) To get $\overline{v_{eq}^2}$ for a two-port:



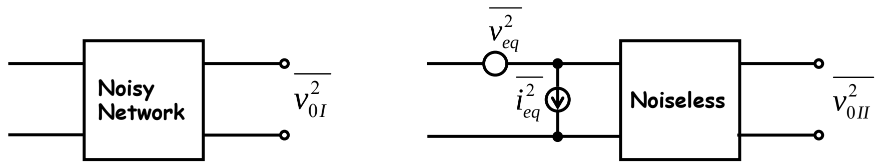
1) Short input, find $\overline{v_{0I}^2}$ (or $\overline{i_{0I}^2}$)
2) For eq. network, short input, find $\overline{v_{0II}^2}$ (or $\overline{i_{0II}^2}$)

$$\overline{v_{0II}^2} = f\left(\overline{v_{eq}^2}\right) \qquad \overline{i_{0II}^2} = f\left(\overline{v_{eq}^2}\right)$$

3) Set $\overline{v_{0I}^2} = \overline{v_{0II}^2}$ → solve for $\overline{v_{eq}^2}$ (or $\overline{i_{0I}^2} = \overline{i_{0II}^2}$)

Calculation of $\overline{v_{eq}^2}$ and $\overline{i_{eq}^2}$ (cont)

b) To get $\overline{i_{eq}^2}$ for a 2-port:



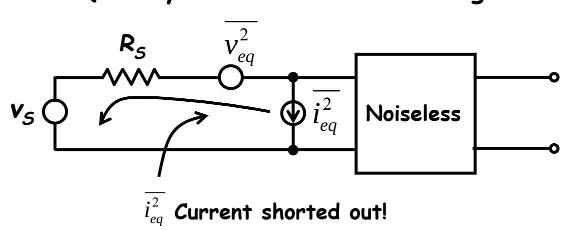
1) Open input, find $\overline{v_{0I}^2}$ (or $\overline{i_{0I}^2}$)
2) Open input for eq. circuit, find $\overline{v_{0II}^2}$ (or $\overline{i_{0II}^2}$)
3) Set $\overline{v_{0I}^2} = \overline{v_{0II}^2}\left(\overline{i_{eq}^2}\right)$ → solve for $\overline{i_{eq}^2}$ (or $\overline{i_{0I}^2} = \overline{i_{0II}^2}\left(\overline{i_{eq}^2}\right)$)

- Once the equivalent input-referred noise generators are found, noise calculations become straightforward as long as the noise generators can be treated as uncorrelated

Cases Where Correlation Is Not Important

- There are two common cases where correlation can be ignored:
  1. Source resistance R_s is small compared to input resistance R_i → i.e., voltage source input
  2. Source resistance R_s is large compared to input resistance R_i → i.e., current source input

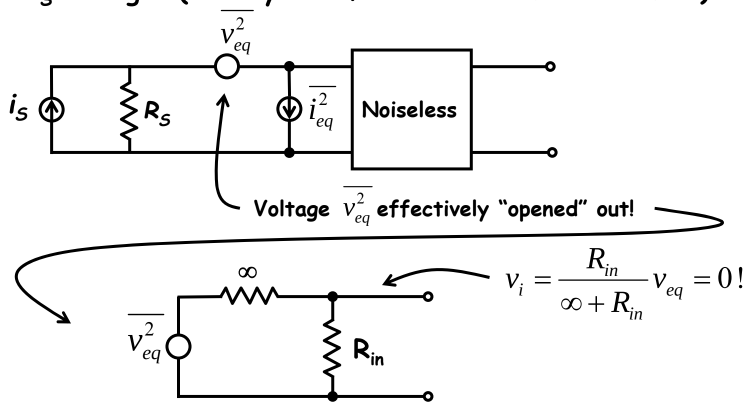
1) R_S = small (ideally = 0 for an ideal voltage source):



∴ For R_S= small, $\overline{i_{eq}^2}$ can be neglected → only $\overline{v_{eq}^2}$ is important!
(Thus, we need not deal with correlation)

Cases Where Correlation Is Not Important

2) R_S = large (Ideally = ∞ for an ideal current source)



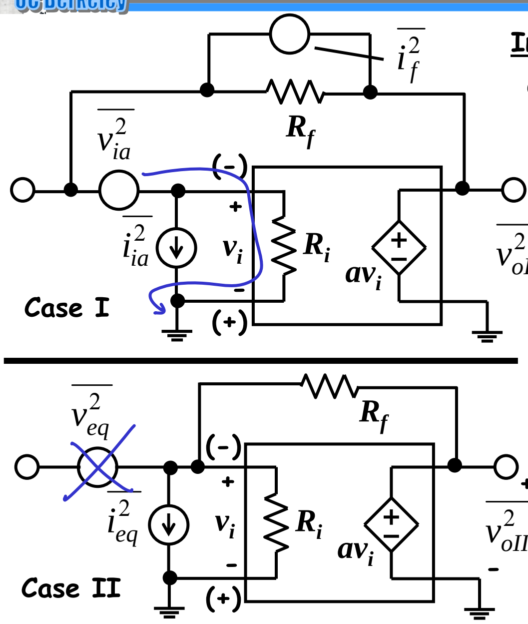
$$v_i = \frac{R_{in}}{\infty + R_{in}} v_{eq} = 0!$$

∴ For R_S= large, $\overline{v_{eq}^2}$ can be neglected!

→ only $\overline{i_{eq}^2}$ is important!

(… and again, we need not deal with correlation)

Example: TransR Amplifier Noise



Input-referred current noise:

Open inputs; equate output voltage noise.

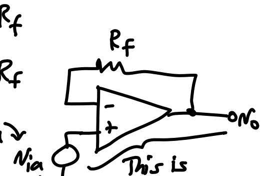
Case I:

$v_{oI1} = i_{ia} R_f$

$v_{oI2} = i_f R_f$

$v_{oI3} = v_{ia}$

Move v_{ia} through Rin to

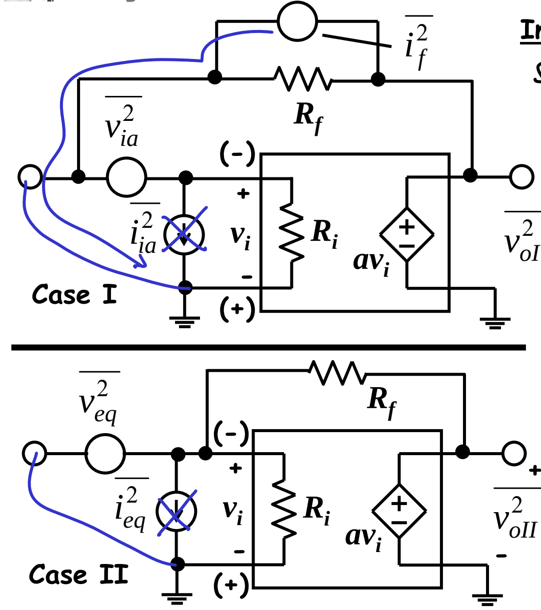


$$\therefore \overline{v_{oI}^2} = \overline{i_{ia}^2} R_f^2 + \overline{i_f^2} R_f^2 + \overline{v_{ia}^2}$$

Case II: $\overline{v_{oII}^2} = \overline{i_{eq}^2} R_f^2$

$$\therefore \boxed{\overline{i_{eq}^2} = \overline{i_{ia}^2} + \overline{i_f^2} + \frac{\overline{v_{ia}^2}}{R_f^2}}$$

Example: TransR Amplifier Noise (cont)



Input-referred voltage noise:

Short inputs; equate output voltage noise

Case I:

$\overline{v_{oI}^2} = \overline{v_{ia}^2} a^2$

(Both $\overline{i_{ia}^2}$ & $\overline{i_f^2}$ are shorted out.)

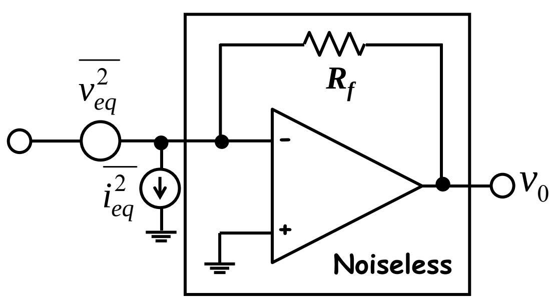
Case II:

$\overline{v_{oII}^2} = \overline{v_{eq}^2} a^2$

$\therefore \boxed{\overline{v_{eq}^2} = \overline{v_{ia}^2}}$

Example: TransR Amplifier Noise (cont)

- To summarize, for a transresistance amplifier, the equivalent input-referred current and voltage noise generators are given by:

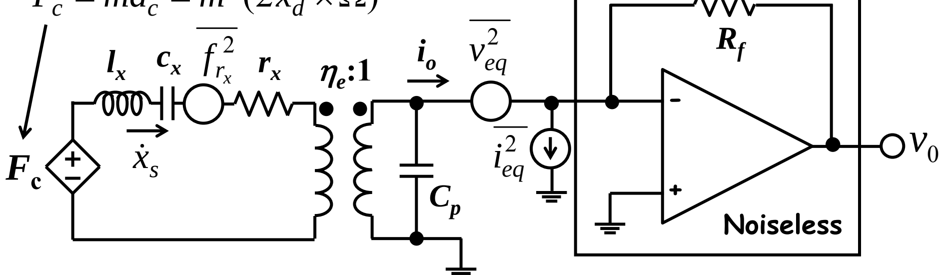


$$\boxed{\overline{i_{eq}^2} = \overline{i_{ia}^2} + \overline{i_f^2} + \frac{\overline{v_{ia}^2}}{R_f^2}} \qquad \boxed{\overline{v_{eq}^2} = \overline{v_{ia}^2}}$$

Back to Gyro Noise & MDS

Example: Gyro MDS Calculation

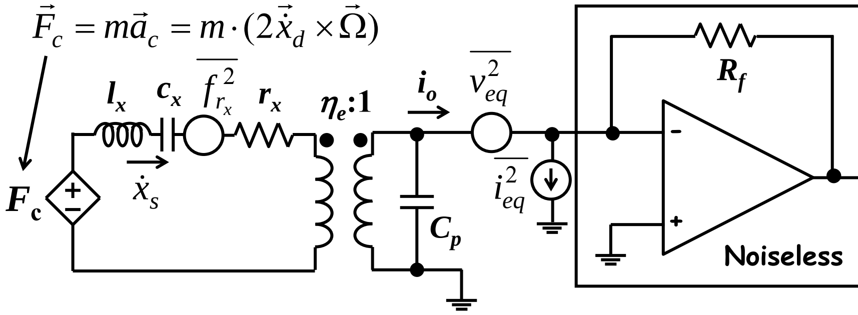
$$\vec{F}_c = m\vec{a}_c = m \cdot (2\vec{\dot{x}}_d \times \vec{\Omega})$$



- The gyro sense presents a large effective source impedance
  - Currents are the important variable; voltages are "opened" out
  - Must compare i_o with the total current noise i_{eqTOT} going into the amplifier circuit

Example: Gyro MDS Calculation (cont)

$$\vec{F}_c = m\vec{a}_c = m\cdot(2\dot{\vec{x}}_d \times \vec{\Omega})$$



- First, find the rotation to i_o transfer function:

$$\dot{x}_s = \frac{\omega_s Q}{k_s}\Theta_s(j\omega_d)F_s = \frac{\omega_s Q}{k_s}\cdot 2\omega_d x_d \Omega m \cdot \Theta(j\omega_d)$$

$$[F_s = F_c = 2\omega_d x_d \Omega m]$$

$$\frac{1}{\omega_s^2}$$

$$\dot{x}_s = 2\frac{\omega_d}{\omega_s}Q x_d \Theta(j\omega_d)\cdot\Omega$$

Example: Gyro MDS Calculation (cont)

$$i_o = \eta_e \dot{x}_s = \underbrace{2\frac{\omega_d}{\omega_s}Q x_d \eta_e \Theta(j\omega_d)}_{A \triangleq \text{scale factor}}\cdot\Omega \rightarrow \boxed{\begin{array}{l} i_o = A\Omega \\ \text{Where } A = 2\frac{\omega_d}{\omega_s}Q x_d \eta_e \Theta(j\omega_d)\end{array}}$$

When $\Omega = \Omega_{min} \triangleq MDS$, $i_o = i_{eqTOT}$ ← input-referred noise current entering the sense amplifier → in pA/$\sqrt{Hz}$

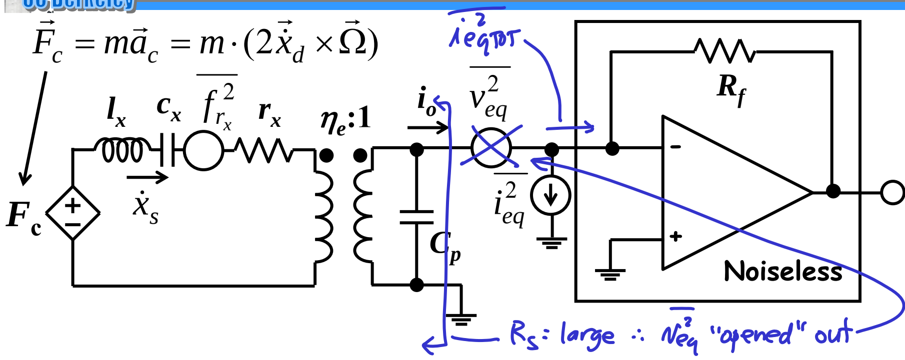
$$\therefore\ i_{eqTOT} = A\Omega_{min} \rightarrow \boxed{\Omega_{min} = \frac{i_{eqTOT}}{A}\left(\frac{3600s}{hr}\right)\left(\frac{180^\circ}{\pi}\right)\ \left[(^\circ/hr)/\sqrt{Hz}\right]}$$

$$\boxed{\text{Angle Random Walk} = ARW = \frac{1}{60}\Omega_{min}\ \left[^\circ/\sqrt{hr}\right]}$$

Easier to determine directional error as a function of elapsed time.

Example: Gyro MDS Calculation (cont)

$$\vec{F}_c = m\vec{a}_c = m\cdot(2\vec{\dot{x}}_d \times \vec{\Omega})$$

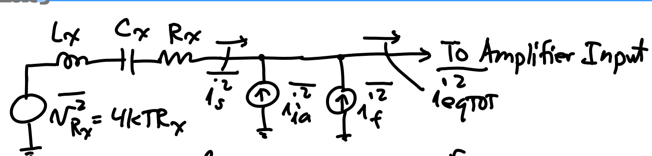


- Now, find the i_{eqTOT} entering the amplifier input:

$$i_{eqTOT} = i_s + i_{eq} \rightarrow \overline{i_{eqTOT}^2} = \overline{i_s^2} + \overline{i_f^2} + \overline{i_{ia}^2} + \frac{\overline{v_{ia}^2}}{R_f^2} \qquad \frac{\overline{f_{r_x}^2}}{\Delta f} = 4kTr_x$$

Brownian motion noise of the sense element → determined entirely by the noise in $r_x \rightarrow \overline{f_{r_x}^2}$
↳ easiest to convert to an all electrical equiv. ckt.

Example: Gyro MDS Calculation (cont)



where $L_x = \frac{l_x}{\eta_e^2}$, $C_x = \eta_e^2 c_x$, $R_x = \frac{r_x}{\eta_e^2}$

$$\therefore\ i_s = v_{R_x}\left(\frac{1}{R_x}\right)\Theta(j\omega_d) \rightarrow \frac{\overline{i_s^2}}{\Delta f} = 4kTR_x\left(\frac{1}{R_x^2}\right)|\Theta(j\omega_d)|^2$$

$$\Rightarrow \boxed{\frac{\overline{i_s^2}}{\Delta f} = \frac{4kT}{R_x}|\Theta(j\omega_d)|^2}$$

Thus:

$$\boxed{\frac{\overline{i_{eqTOT}^2}}{\Delta f} = \frac{4kT}{R_x}|\Theta(j\omega_d)|^2 + \frac{4kT}{R_f} + \frac{\overline{i_{ia}^2}}{\Delta f} + \frac{\overline{v_{ia}^2}}{\Delta f}\left(\frac{1}{R_f^2}\right)}$$

Learn to get these from EE240.
↳ or just get them from a data sheet ...

LF356 Op Amp Data Sheet

UC Berkeley

LF155/LF156/LF256/LF257/LF355/LF356/LF357
JFET Input Operational Amplifiers

General Description

These are the first monolithic JFET input operational amplifiers to incorporate well matched, high voltage JFETs on the same chip with standard bipolar transistors (BI-FET™ Technology). These amplifiers feature low input bias and offset currents/low offset voltage and offset voltage drift, coupled with offset adjust which does not degrade drift or common-mode rejection. The devices are also designed for high slew rate, wide bandwidth, extremely fast settling time, low voltage and current noise and a low 1/f noise corner.

- Logarithmic amplifiers
- Photocell amplifiers
- Sample and Hold circuits

Common Features
- Low input bias current: 30pA
- Low Input Offset Current: 3pA
- High input impedance: $10^{12}\Omega$
- Low input noise current: 0.01 pA/$\sqrt{\text{Hz}}$
- High common-mode rejection ratio: 100 dB
- Large dc voltage gain: 106 dB

$\sqrt{\frac{\overline{i_{ia}^2}}{\Delta f}} = 0.01\,\text{pA}/\sqrt{\text{Hz}}$

Features

Advantages
- Replace expensive hybrid and module FET op amps
- Rugged JFETs allow blow-out free handling compared with MOSFET input devices
- Excellent for low noise applications using either high or low source impedance—very low 1/f corner
- Offset adjust does not degrade drift or common-mode rejection as in most monolithic amplifiers
- New output stage allows use of large capacitive loads (5,000 pF) without stability problems
- Internal compensation and large differential input voltage capability

Applications
- Precision high speed integrators
- Fast D/A and A/D converters
- High impedance buffers
- Wideband, low noise, low drift amplifiers

Uncommon Features

	LF155/ LF355	LF156/ LF256/ LF356	LF257/ LF357 (A_V=5)	Units
Extremely fast settling time to 0.01%	4	1.5	1.5	μs
Fast slew rate	5	12	50	V/μs
Wide gain bandwidth	2.5	5	20	MHz
Low input noise voltage	20	12	12	nV/$\sqrt{\text{Hz}}$

$\sqrt{\frac{\overline{v_{ia}^2}}{\Delta f}} = 12\,\text{nV}/\sqrt{\text{Hz}}$

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Example ARW Calculation

UC Berkeley

- Example Design:
  - Sensor Element:

    m = (100μm)(100μm)(20μm)(2300kg/m³) = 4.6x10⁻¹⁰kg

    ω_s = 2π(15kHz)

    ω_d = 2π(10kHz)

    $k_s = \omega_s^2 m$ = 4.09 N/m

    x_d = 20 μm

    Q_s = 50,000

    V_P = 5V

    h = 20 μm

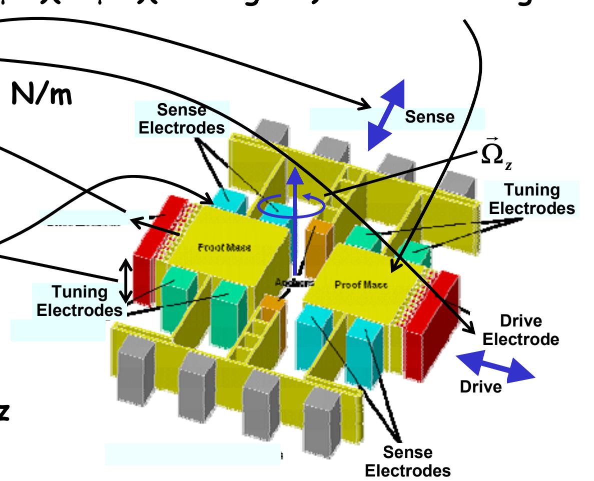
    d = 1 μm

  - Sensing Circuitry:

    R_f = 100kΩ

    i_{ia} = 0.01 pA/√Hz

    v_{ia} = 12 nV/√Hz



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Example ARW Calculation (cont)

Get rotation rate to output current scale factor:

$$A = 2\frac{\omega_d}{\omega_s}Q_s x_d \eta_e |\Theta(j\omega_d)| = 2\left(\frac{10k}{15k}\right)(50k)(20\mu)(5)(2000\epsilon_o)(0.000024) = \underline{2.83\times10^{-12}C}$$

$$\Theta(j\omega_d) = \frac{(j\omega_d)(\omega_s/Q_s)}{-\omega_d^2 + \frac{j\omega_d\omega_s}{Q_s} + \omega_s^2} = \frac{j(10k)(15k)/(50k)}{(15k)^2-(10k)^2 + \frac{j(10k)(15k)}{50k}} = \frac{j(3k)}{1.25\times10^8 + j(3k)}$$

$$\rightarrow |\Theta(j\omega_d)| = \frac{3k}{\sqrt{(1.25\times10^8)^2 + (3k)^2}} = 0.000024$$

$$\frac{\partial C}{\partial x} = \frac{C_o}{d} = \frac{\epsilon_o h W_p}{d} = \frac{\epsilon_o(20\mu)(100\mu)}{(1\mu)^2} = 2000\epsilon_o \rightarrow \eta_e = V_P\frac{\partial C}{\partial x} = \overbrace{5(2000\epsilon_o)}^{8.854\times10^{-8}F/m}$$

Assume electrode covers the whole sidewall.

ϵ_o: 8.854×10^{-12} F/m

Then, get noise:

$$\frac{\overline{i_{eqTOT}^2}}{\Delta f} = \frac{4kT}{R_x}|\Theta(j\omega_d)|^2 + \frac{4kT}{R_f} + \frac{\overline{i_{ia}^2}}{\Delta f} + \frac{\overline{v_{ia}^2}}{\Delta f}\left(\frac{1}{R_f^2}\right)$$

Example ARW Calculation (cont)

$$R_x = \frac{\omega_s m}{Q_s\eta_e^2} = \frac{2\pi(15k)(4.6\times10^{-10})}{(50k)(8.854\times10^{-8})^2} = 110.6k\Omega$$

$$\frac{\overline{i_{eqTOT}^2}}{\Delta f} = \underbrace{\frac{(1.66\times10^{-20})}{(110.6k)}(0.000024)^2}_{8.64\times10^{-35}A^2/Hz} + \underbrace{\frac{(1.66\times10^{-20})}{1M}}_{1.66\times10^{-26}A^2/Hz} + \underbrace{(0.01p)^2}_{1\times10^{-28}A^2/Hz} + \underbrace{\frac{(12n)^2}{(1M)^2}}_{1.44\times10^{-28}A^2/Hz}$$

sensor element noise insignificant

Noise from R_f dominates!

$$\therefore \frac{\overline{i_{eqTOT}^2}}{\Delta f} = 1.68\times10^{-26}A^2/Hz \longrightarrow i_{eqTOT} = \sqrt{\frac{\overline{i_{eqTOT}^2}}{\Delta f}} = 1.30\times10^{-13}A/\sqrt{Hz}$$

$$\therefore \Omega_{min} = \frac{i_{eqTOT}}{A}\left(\frac{3600s}{hr}\right)\left(\frac{180^\circ}{\pi}\right) = \frac{1.30\times10^{-13}}{2.83\times10^{-12}}(3600)\left(\frac{180}{\pi}\right) = \underline{9448\ (^\circ/hr)/\sqrt{Hz}}$$

And finally:

$$ARW = \frac{1}{60}\Omega_{min} = \frac{1}{60}(9448) = 157\ ^\circ/\sqrt{hr} = ARW$$

→ Almost turned around in 1 hour!

What if $\omega_d = \omega_s$?

If $\omega_d = \omega_s = 15\text{KHz}$, then $|\Theta(j\omega_d)| = 1$ and

$$A = 2\frac{\omega_d}{\omega_s} Q_s x_d \eta_e |\Theta(j\omega_d)| = 2 Q_s x_d \eta_e = 2(50k)(20\mu)(5)(2000\epsilon_0) = \underline{1.77\times10^{-7}C}$$

$$\frac{\overline{i^2_{eqTOT}}}{\Delta f} = \underbrace{\frac{(1.66\times10^{-29})}{(110.6K)}(1)^2}_{1.51\times10^{-25}A^2/Hz} + \underbrace{\frac{(1.66\times10^{-29})}{1M}}_{1.66\times10^{-26}A^2/Hz} + \underbrace{(0.01p)^2}_{1\times10^{-28}A^2/Hz} + \underbrace{\frac{(12n)^2}{(1M)^2}}_{1.44\times10^{-28}A^2/Hz}$$

Now, the sensor element dominates!

$$\therefore \frac{\overline{i^2_{eqTOT}}}{\Delta f} = 1.67\times10^{-25}A^2/Hz \longrightarrow i_{eqTOT} = \sqrt{\frac{\overline{i^2_{eqTOT}}}{\Delta f}} = 4.08\times10^{-13}A/\sqrt{Hz}$$

$$\therefore \Omega_{min} = \frac{i_{eqTOT}}{A}\left(\frac{3600s}{hr}\right)\left(\frac{180^\circ}{\pi}\right) = \frac{4.08\times10^{-13}}{1.77\times10^{-7}}(3600)\left(\frac{180}{\pi}\right) = \underline{0.476\ (^\circ/hr)/\sqrt{Hz}}$$

And finally:

$$ARW = \frac{1}{60}\Omega_{min} = \frac{1}{60}(0.476) = 0.0079\ ^\circ/\sqrt{hr} = ARW$$

$\Rightarrow$ Navigation grade!