Suppose a step function voltage $V_A$ was suddenly applied across the anchors of a 2 $\mu$m thick polysilicon beam and proof mass setup as shown in Figures PS1.3-1 and PS1.3-2, which also provide lateral dimensions. For polysilicon, assume the following material properties: Young’s modulus $E = 150$ GPa, density $\rho = 2300$ kg/m$^3$, Poisson ratio $\nu = 0.226$, sheet resistance (resistivity*thickness$^{-1}$) = 10 $\Omega$/square, specific heat = 0.77 $J/(g*K)$, and thermal conductivity = 30 W/(m*K).

(a) With what time constant will the proof mass reach its steady-state temperature after the voltage $V_A$ steps from 0V to 1V? Give a formula and a numerical answer with units.

(b) If the final step function value of $V_A$ is 1V, what is the steady-state temperature of the proof mass? Give a formula and a numerical answer with units.

(c) What effect do you think the applied voltage has on the resonance frequency of the structure in the z-direction (into the page)? Give a brief qualitative explanation.
1. Find the electrical circuit:

\[ R_1 = 12 \times 10^{-2} \Omega = 120 \Omega \]
\[ R_2 = 120 \Omega \]
\[ R_3 = 60 \Omega \]
\[ R_4 = 10 \Omega \]

\[ I = \frac{V_A}{R_1 + R_2 + R_3 + R_4} = \frac{IV}{130 \Omega} = 7.3 \text{ mA} \]

Powers:
\[ P_1 = \left(\frac{I}{2}\right)^2 R_1 = 1.78 \text{ mW} \]
\[ P_2 = \left(\frac{I}{3}\right)^2 R_2 = 1.78 \text{ mW} \]
\[ P_3 = I^2 R_3 = 3.56 \text{ mW} \]
\[ P_4 = 0.58 \text{ mW} \]

All \( P_1 - P_4 \) will be power inputs to the thermal circuit.

2. Find the thermal circuit:

\[ R_{th1} = \frac{l}{kA} = \frac{60 \mu}{(30)(24 \mu)(5 \mu)} = 0.2 \text{ MK/W} \]

\[ R_{th2} = R_{th1} = 0.2 \text{ MK/W} \]

\[ R_{th3} = 0.1 \text{ MK/W} \]

\[ R_{th4} = \frac{60 \mu}{(30)(60 \mu)(2 \mu)} = 16 \text{ kK/W} \]

\[ C_{th1} = \rho CV = (2300)(770)(24 \mu)(60 \mu) = 1.06 \text{ nJ/K} \]

\[ C_{th2} = C_{th1} = 1.06 \text{ nJ/K} \]

\[ C_{th3} = 0.53 \text{ nJ/K} \]

\[ C_{th4} = 12.75 \text{ nJ/K} \]
3. Simplify the thermal circuit w/ appropriate assumptions:

\[ \begin{align*}
C_{th1} &= 1.06 \text{ nJ/K} \\
C_{th2} &= 1.06 \text{ nJ/K} \\
C_{th3} &= 0.53 \text{ nJ/K} \\
C_{th4} &= 12.75 \text{ nJ/K}
\end{align*} \]

\[ \begin{align*}
R_{th1} &= 200 \text{ kK/W} \\
R_{th2} &= 200 \text{ kK/W} \\
R_{th3} &= 100 \text{ kK/W} \\
R_{th4} &= 16 \text{ kK/W}
\end{align*} \]

neglect \( C_{th1}, C_{th2}, C_{th3} \)

\[ \text{Simplified steady-state circuit} \]
(a) Time constant circuit: kill all independent sources, i.e. short-circuit voltage sources and open-circuit current sources.

\[ T = \left( \frac{R_{th1}}{2} \parallel R_{th3} \right) C_{th4} = 0.6375 \text{ ms} \]
\[ = 637.5 \mu\text{s} \]

(b) Apply superposition: (at steady-state all capacitors are fully-charged/open)

1. Kill \( P_3, P_4 \):

\[ \Delta T_{pm1} = 2 P_1 \frac{R_{th1}}{R_{th1} + R_{th3}} \]
\[ R_{th3} = 89 \text{ K} \]

2. Kill \( P_2, P_3 \):

\[ \Delta T_{pm2} = \left( R_{th3} \parallel \frac{R_{th1}}{2} \right) P_4 = 29.5 \text{ K} \]
\[ \uparrow \quad \text{even } P_4 \text{ could be neglected!} \]

3. Kill \( P_2, P_4 \):

\[ \Delta T_{pm3} = P_3 \frac{R_{th3}^2}{R_{th3} + R_{th3} + \frac{R_{th1}}{2}} \]
\[ \frac{R_{th1}}{2} = 89 \text{ K} \]

\[ \text{In total } \Rightarrow T_{pm} = T_0 + \sum_{k=1}^{3} \Delta T_{pmk} = 298 \text{ K} + 89 \text{ K} + 29.5 \text{ K} + 89 \text{ K} = 506.5 \text{ K} \]
*What happens if we neglect Py?

$$T_{pm} = 298K + 89K + 89K = 476K$$

$$\epsilon_m = 100 \left( 1 - \frac{T_{pm'}}{T_{pm}} \right) = 100 \cdot 5.84\% < 10\%$$

so you're still fine!

(c) Joule heating causes polysilicon to expand but since the anchors are fixed, compressive axial stress results in the beams. This lowers the beam stiffness and thus lowers the resonance frequency: $$\omega_o = \sqrt{\frac{k}{m}}$$