## PROBLEM SET \#1A

Issued: Tuesday, Jan. 19, 2016
Due: Wednesday, Feb. 3, 2016, at 8:00 a.m. in the EE C247B homework box near 125 Cory.
This homework assignment is intended to give you some early practice playing with dimensions and exploring how scaling can greatly improve or degrade certain performance characteristics of mechanical systems. Don't worry at this point if you do not understand fully some of the physical expressions used. They will be revisited later in the semester. This assignment just gives you a chance to play with them a bit.
Use the material parameters given in Table PS1A. 1 wherever needed.

| TYPE | SILICON <br> NITRIDE | INTERCONNECT <br> POLYSILICON | STRUCTURAL <br> POLYSILICON | UNIT |
| :---: | :---: | :---: | :---: | :---: |
| DENSITY | 3200 | 2300 | 2300 | $\mathrm{~kg} / \mathrm{m}^{3}$ |
| YOUNG'S <br> MODULUS | 200 | 150 | 150 | GPa |
| POISSON <br> RATIO | 0.280 | 0.226 | 0.226 | - |
| ELECTRICAL <br> RESISTIVITY | $10^{21}$ | $10^{-5}$ | $10^{-5}$ | $\Omega . \mathrm{m}$ |
| THERMAL | 43 | 30 | 30 | $\mathrm{~W} / \mathrm{m} . \mathrm{K}$ |
| CONDUCTIVITY <br> SPECIFIC <br> HEAT | 1.10 | 0.77 | 0.77 | $\mathrm{~J} / \mathrm{g} . \mathrm{K}$ |
| MICRO-OVEN <br> THICKNESS | 1.0 | 0.4 | 2.0 | $\mu \mathrm{~m}$ |

Table PS1A. 1

1. Suppose you are asked to design a polysilicon beam resonator, such as discussed in lecture and given below in Fig. PS1A.1.
(a) The general equation for the deflection of this beam is given by the Euler-Bernoulli equation as follows

$$
\begin{equation*}
\frac{\partial^{4} v(y)}{\partial y^{4}}=\omega^{2} \frac{\rho S}{E I} v(y) \tag{1}
\end{equation*}
$$

where $v$ is the displacement function, $y$ is the planar spatial coordinate as shown in Fig. PS1A.1(b), $\omega$ is the resonance frequency in radians, $S$ is the cross-section area, and $I$ is the moment of inertia of the beam cross-section. Derive the general solution of the EulerBernoulli equation, i.e. find the most general solution of $v(y)$. (Note that your answer will have four constants.)


Fig. PS1A. 1
(b) The beam given in Fig. PS1A. 1 is a generic one with no end conditions specified. However, resonance frequency and mode shape determination requires the knowledge of end conditions. Some common end conditions are listed in Table PS1A. 2 below.

TYPE $\quad 1^{\text {ST }}$ BOUNDARY CONDITION $\quad 2^{\text {ND }}$ BOUNDARY CONDITION

| FIXED | $v=0$ | No deflection <br> at the <br> boundary | $\frac{\partial v}{\partial y}=0$ | Horizontal at <br> the boundary |
| :---: | :---: | :---: | :---: | :---: |
| FREE | $\frac{\partial^{2} v}{\partial y^{2}}=0$ | No bending <br> moment | $\frac{\partial^{3} v}{\partial y^{3}}=0$ | No shearing <br> force |
| SIMPLY | $v=0$ | No deflection <br> at the <br> boundary | $\frac{\partial^{2} v}{\partial y^{2}}=0$ | No bending <br> moment |

Table PS1A. 2

Find an expression for the resonance frequencies of the first three modes for the beam resonators with different end conditions given in Fig. PS1A. 2 below. Don't worry at this point on how a free-free beam is actually fixed to the substrate.


Fig. PS1A. 2
(c) Find the first mode shape of cantilever, fixed-fixed beam, and free-free beam resonators and plot them on the same graph. Do not forget to normalize the mode shapes. Normalization in this context means setting the peak amplitude to unity.
(d) Assuming $H=2 \mu \mathrm{~m}$ and $W=10 \mu \mathrm{~m}$ design a free-free beam resonator at (i) 10 MHz , (ii) 100 MHz , and (iii) 1 GHz .
(e) Assuming $H=2 \mu \mathrm{~m}$ and $W=10 \mu \mathrm{~m}$, plot $f_{0}$ vs. $L / H$ for a free-free beam with $L / H$ going from 1 to 40. Make sure the step size is small enough to obtain a smooth curve.
(f) Euler-Bernoulli theory is actually not very accurate when the length of the beam begins to approach its thickness, mainly because it ignores shear displacements and rotary inertias. (These are things that you will learn more about later in the course.) For cases where thickness approaches length, the more complicated Timoshenko design procedure should be used to model a beam's resonance characteristics. For a free-free beam, Timoshenko's design procedure uses the following equation:

$$
\begin{equation*}
\tan \frac{\beta}{2}+\frac{\alpha}{\beta}\left(\frac{\alpha^{2}+g^{2}}{\beta^{2}-g^{2}}\right) \tanh \frac{\alpha}{2}=0 \tag{2}
\end{equation*}
$$

where

$$
\begin{equation*}
g^{2}=\omega^{2} L^{2}\left(\frac{\rho}{E}\right) \tag{3}
\end{equation*}
$$

and

$$
\left.\begin{array}{l}
\beta^{2}  \tag{4}\\
\alpha^{2}
\end{array}\right\}=\frac{g^{2}}{2}\left[ \pm\left(1+\frac{E}{\kappa G}\right)+\sqrt{\left(1-\frac{E}{\kappa G}\right)^{2}+\frac{4 L^{2} H W}{g^{2} I}}\right]
$$

where $\kappa$ is the shear-deflection coefficient (for a rectangular cross-section, $\kappa$ is $2 / 3$ ) and $G$ is the shear modulus of elasticity given as

$$
\begin{equation*}
G=\frac{E}{2(1+v)} \tag{5}
\end{equation*}
$$

Use Timoshenko's formulas above to determine the actual frequencies of the beams designed in part (d) above and determine the percent mismatch. Plot $f_{0}$ vs. $L / H$ curves obtained in part (e) with Euler-Bernoulli and Timoshenko theories on the same graph. Determine the critical $L / H$ aspect ratio after which the mismatch between the two theories is less than $5 \%$. The percent mismatch is defined as follows

$$
\begin{equation*}
\text { Percent Mismatch }=100\left(1-\frac{f_{\text {Timoshenko }}}{f_{\text {Euler-Bernoulli }}}\right) \tag{6}
\end{equation*}
$$

where $f_{\text {Euler-Bernoulli }}$ is the resonance frequency obtained with Euler-Bernoulli theory and $f_{\text {Timoshenko }}$ is the resonance frequency obtained with Timoshenko theory.


Fig. PS1A. 3
(g) Assume that a $150 \mathrm{~mm}\left(6^{\prime \prime}\right)$ diameter wafer has a useful area of $100 \mathrm{~mm} \times 100 \mathrm{~mm}$ over which resonators can be fabricated as shown in Fig. PS1A.3. (Here, the edges of the wafer are for handling, so do not yield working devices.) A dicing saw is used to cut the wafer into individual dies and the width of each cut is $50 \mu \mathrm{~m}$. Each sensor requires a square unit cell with a minimum area of $9 L^{2}$. The cost per sensor is given by

$$
\begin{equation*}
C(n, d)=\frac{\$ 3000+\$ 1 \times n}{d}+\$ 2 \tag{7}
\end{equation*}
$$

where $n$ is the number of cuts through the wafer and $d$ is the number of dies. Here, the fixed $\$ 2$ cost per sensor is due to post processing, packaging, and testing costs. Assuming that the minimum die size that can be reliably handled is $1 \mathrm{~mm} \times 1 \mathrm{~mm}$. What is the lowest achievable fabrication cost per sensor (to the nearest cent) and what is the corresponding maximum resonator length (to the nearest ten microns)? [Hint: it would be helpful to define $d(n)$ and to find $n$.]

