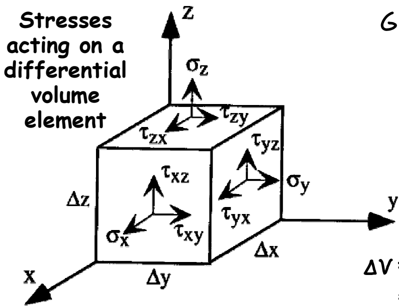
Volume Change for a Uniaxial Stress

Stresses acting on a differential volume element



Given an x-directed uniaxial stress, σ_x:

$$\Delta x \rightarrow \Delta x(1+\epsilon_x)$$
$$\Delta y \rightarrow \Delta y(1-\nu\epsilon_x)$$
$$\Delta z \rightarrow \Delta z(1-\nu\epsilon_x)$$

$\downarrow$ The resulting change in volume ΔV

$$\Delta V = \Delta x \Delta y \Delta z(1+\epsilon_x)(1-\nu\epsilon_x)^2 - \Delta x \Delta y \Delta z$$
$$= \Delta x \Delta y \Delta z\left[(1+\epsilon_x)(1-\nu\epsilon_x)^2 - 1\right]$$

[Assume small strains] $\Rightarrow \Delta V = \Delta x \Delta y \Delta z\left[(1+\epsilon_x)(1-2\nu\epsilon_x) - 1\right]$

$[(1+mx)^n \approx 1+nmx] \Rightarrow \approx \Delta x \Delta y \Delta z\left[\cancel{1} + \epsilon_x - 2\nu\epsilon_x - \underbrace{\cancel{2\nu\epsilon_x^2}}_{\text{negl.}} - \cancel{1}\right]$

$$\boxed{\Delta V \approx \Delta x \Delta y \Delta z(1-2\nu)\epsilon_x}$$

For $\nu = 0.5$ (rubber) $\rightarrow$ no ΔV!
$\nu < 0.5 \rightarrow$ finite ΔV

Isotropic Elasticity in 3D

- Isotropic = same in all directions
- The complete stress-strain relations for an isotropic elastic solid in 3D: (i.e., a generalized Hooke's Law)

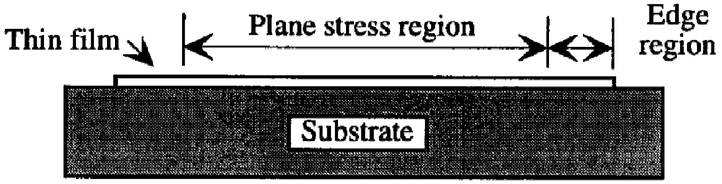
$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right] \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \underbrace{\nu\left(\sigma_x + \sigma_y\right)}_{\text{Basically, add in off-axis strains from normal stresses in other directions}}\right] \qquad \gamma_{zx} = \frac{1}{G}\tau_{zx}$$

Important Case: Plane Stress

- **Common case**: very thin film coating a thin, relatively rigid substrate (e.g., a silicon wafer)



- At regions more than 3 thicknesses from edges, the top surface is stress-free → $\sigma_z = 0$
- Get two components of in-plane stress:

$$\varepsilon_x = (1/E)[\sigma_x - \nu(\sigma_y + 0)]$$
$$\varepsilon_y = (1/E)[\sigma_y - \nu(\sigma_x + 0)]$$

Important Case: Plane Stress (cont.)

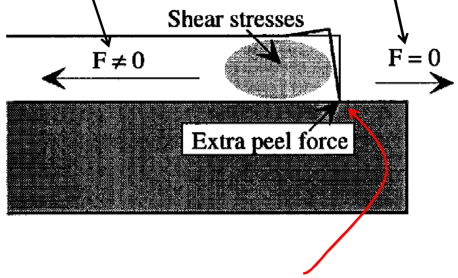
- Symmetry in the xy-plane → $\sigma_x = \sigma_y = \sigma$
- Thus, the in-plane strain components are: $\varepsilon_x = \varepsilon_y = \varepsilon$ where

$$\varepsilon_x = (1/E)[\sigma - \nu\sigma] = \frac{\sigma}{[E/(1-\nu)]} = \frac{\sigma}{E'}$$

and where

$$\text{Biaxial Modulus} \triangleq E' = \frac{E}{1-\nu}$$

Edge Region of a Tensile (σ>0) Film



Net non-zero in-plane force (that we just analyzed)

At free edge, in-plane force must be zero

Film must be bent back, here

There's no Poisson contraction, so the film is slightly thicker, here

Discontinuity of stress at the attached corner → stress concentration

Peel forces that can peel the film off the surface

Linear Thermal Expansion

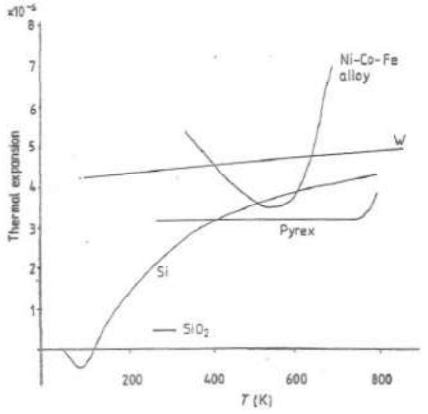
- As temperature increases, most solids expand in volume
- **Definition**: linear thermal expansion coefficient

$$\left.\begin{array}{c}\text{Linear thermal}\\ \text{expansion coefficient}\end{array}\right\} \triangleq \alpha_T = \frac{d\varepsilon_x}{dT} \qquad [\text{Kelvin}^{-1}]$$

Remarks:

- α_T values tend to be in the 10^{-6} to 10^{-7} range
- Can capture the 10^{-6} by using dimensions of μstrain/K, where 10^{-6} K^{-1} = 1 μstrain/K
- In 3D, get volume thermal expansion coefficient ⟶ $\frac{\Delta V}{V} = 3\alpha_T \Delta T$
- For moderate temperature excursions, α_T can be treated as a constant of the material, but in actuality, it is a function of temperature

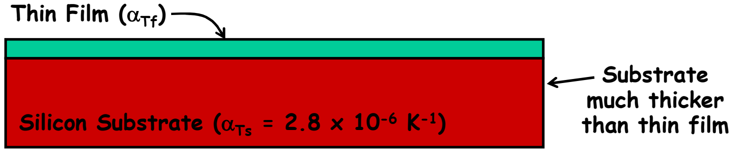
α_T As a Function of Temperature



[Madou, Fundamentals of Microfabrication, CRC Press, 1998]

- Cubic symmetry implies that α is independent of direction

Thin-Film Thermal Stress



- Assume film is deposited stress-free at a temperature T_r, then the whole thing is cooled to room temperature T_r
- Substrate much thicker than thin film → substrate dictates the amount of contraction for both it and the thin film

Thermal strain of the substrate: (in one in-plane dimension)

$$\epsilon_s = -\alpha_{Ts}\Delta T \quad , \text{ where } \Delta T = T_d - T_r$$

If the film were not attached to the substrate: $\epsilon_{f,free} = -\alpha_{Tf}\Delta T$

over

Linear Thermal Expansion

But the film is attached to the substrate, so the actual strain in the film is the same as that in the substrate:

$$\epsilon_{f,attached} = -\alpha_{Ts}\Delta T$$

Thus:

$$\boxed{\text{Thermal Mismatch Strain} = \epsilon_{f,mismatch} = (\alpha_{Tf} - \alpha_{Ts})\Delta T}$$

↳ Note that this is biaxial strain

↳ it can only be developed by an in-plane biaxial stress:

$$\sigma_{f,mismatch} = \left(\frac{E}{1-\nu}\right)\epsilon_{f,mismatch}$$

Ex. Thin-film is polyimide → $\alpha_{Tf} = 70\times10^{-6}\,K^{-1}$, $E' = 4\,GPa$
deposited @ 250°C, then cooled to RT = 25°C → $\Delta T = 225\,K$

e.g., SiO_2

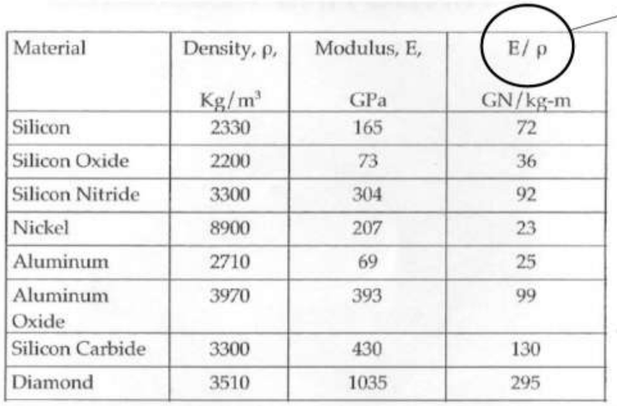
$$\epsilon_{f,mismatch} = (70-2.8)\mu(225) = 1.5\times10^{-2}$$

$$\sigma_{f,mismatch} = (4)(1.5\times10^{-2}) = \underline{\underline{60.5\,MPa}}$$

← stress is (+), ∴ tensile
[(-) would be compressive]

MEMS Material Properties

Material Properties for MEMS



Material	Density, ρ, Kg/m³	Modulus, E, GPa	E/ρ GN/kg-m
Silicon	2330	165	72
Silicon Oxide	2200	73	36
Silicon Nitride	3300	304	92
Nickel	8900	207	23
Aluminum	2710	69	25
Aluminum Oxide	3970	393	99
Silicon Carbide	3300	430	130
Diamond	3510	1035	295

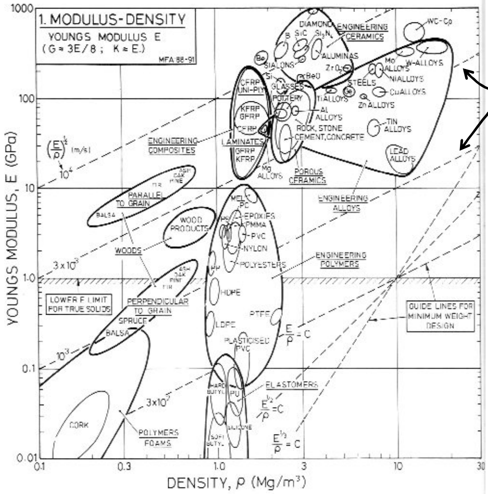
Units: $(m/s)^2$

↓

$\sqrt{(E/\rho)}$ is acoustic velocity

[Mark Spearing, MIT]

Young's Modulus Versus Density

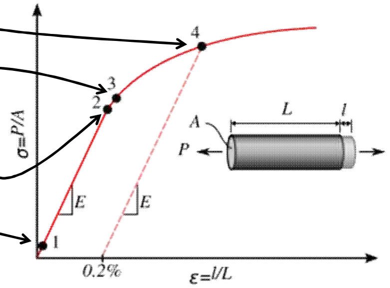


Lines of constant acoustic velocity

[Ashby, Mechanics of Materials, Pergamon, 1992]

Yield Strength

- Definition: the stress at which a material experiences significant plastic deformation (defined at 0.2% offset pt.)
- Below the yield point: material deforms elastically → returns to its original shape when the applied stress is removed
- Beyond the yield point: some fraction of the deformation is permanent and non-reversible



Yield Strength: defined at 0.2% offset pt.

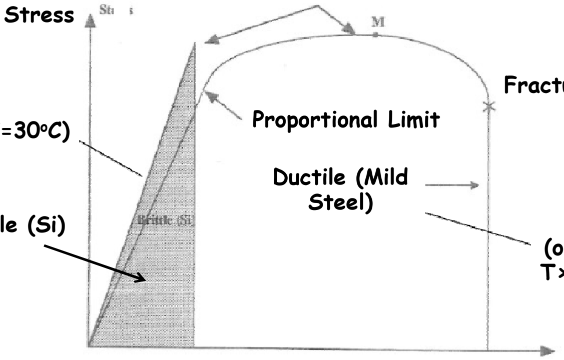
Elastic Limit: stress at which permanent deformation begins

Proportionality Limit: point at which curve goes nonlinear

True Elastic Limit: lowest stress at which dislocations move

Yield Strength (cont.)

- Below: typical stress vs. strain curves for brittle (e.g., Si) and ductile (e.g. steel) materials



Stress

Tensile Strength

Fracture

Proportional Limit

(Si @ T=30°C)

Ductile (Mild Steel)

Brittle (Si)

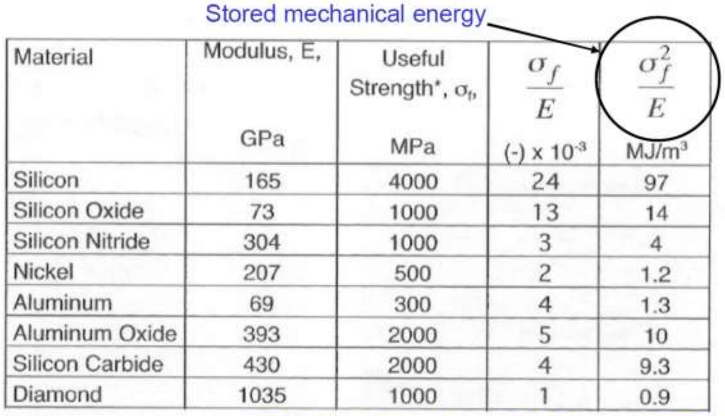
(or Si @ T>900°C)

[Maluf]

Strain

Young's Modulus and Useful Strength

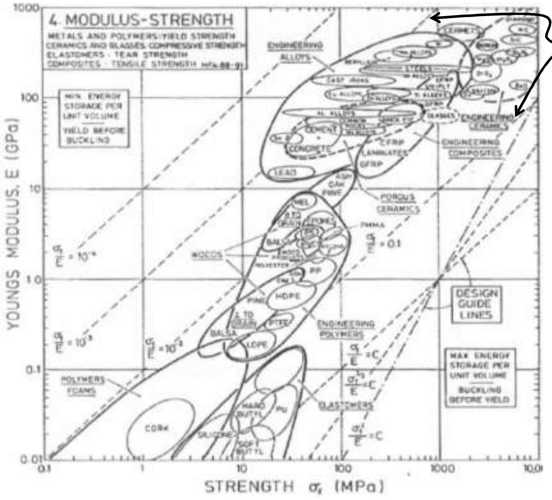
Stored mechanical energy



Material	Modulus, E, GPa	Useful Strength*, σ_f, MPa	$\frac{\sigma_f}{E}$ (-) x 10^{-3}	$\frac{\sigma_f^2}{E}$ MJ/m³
Silicon	165	4000	24	97
Silicon Oxide	73	1000	13	14
Silicon Nitride	304	1000	3	4
Nickel	207	500	2	1.2
Aluminum	69	300	4	1.3
Aluminum Oxide	393	2000	5	10
Silicon Carbide	430	2000	4	9.3
Diamond	1035	1000	1	0.9

From Mark Spearing, MIT, *Future of MEMS Workshop*, Cambridge, England, May 2003

Young's Modulus Versus Strength

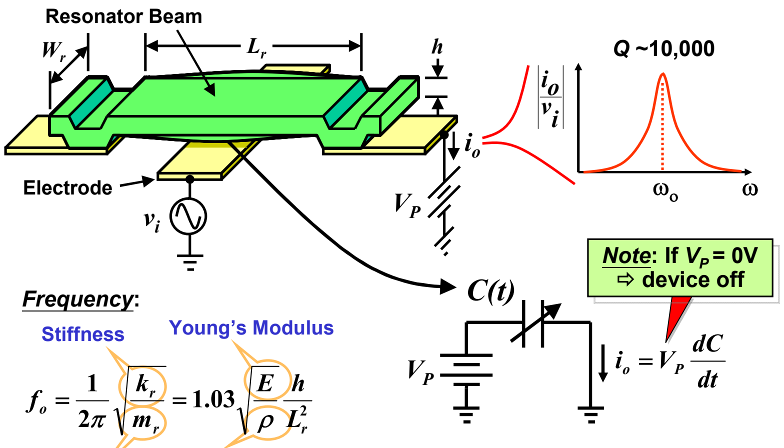


Lines of constant maximum strain

[Ashby, Mechanics of Materials, Pergamon, 1992]

Quality Factor (or Q)

Clamped-Clamped Beam μResonator



Note: If V_P = 0V ⇨ device off

Frequency:

Stiffness　Young's Modulus

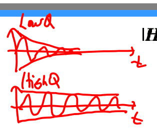
$$f_o = \frac{1}{2\pi}\sqrt{\frac{k_r}{m_r}} = 1.03\sqrt{\frac{E}{\rho}}\frac{h}{L_r^2}$$

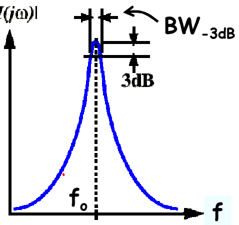
Density

Mass → (e.g., $m_r = 10^{-13}$ kg) → Smaller mass ⇨ higher freq. range and lower series R_x

Quality Factor (or Q)

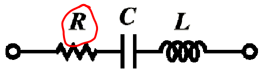
- Measure of the frequency selectivity of a tuned circuit
- Definition:





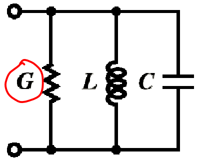
$$Q = \frac{\text{Total Energy Per Cycle}}{\text{Energy Lost Per Cycle}} = \frac{f_o}{BW_{3dB}}$$

- Example: series LCR circuit



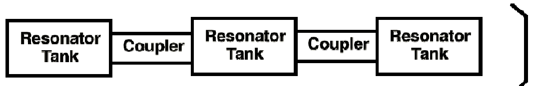
$$\Rightarrow\ Q = \frac{\text{Im}(Z)}{\text{Re}(Z)} = \frac{\omega_o L}{R} = \frac{1}{\omega_o CR}$$

- Example: parallel LCR circuit



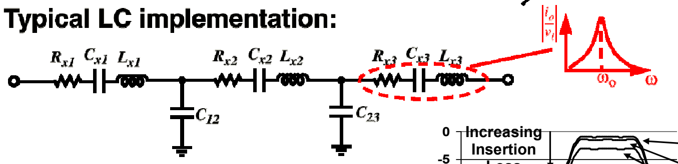
$$\Rightarrow\ Q = \frac{\text{Im}(Y)}{\text{Re}(Y)} = \frac{\omega_o C}{G} = \frac{1}{\omega_o LG}$$

Selective Low-Loss Filters: Need Q



General BPF Implementation

Typical LC implementation:



- In resonator-based filters: high tank Q ⇔ low insertion loss
- At right: a 0.1% bandwidth, 3-res filter @ 1 GHz (simulated)
  ⇘ heavy insertion loss for resonator Q < 10,000