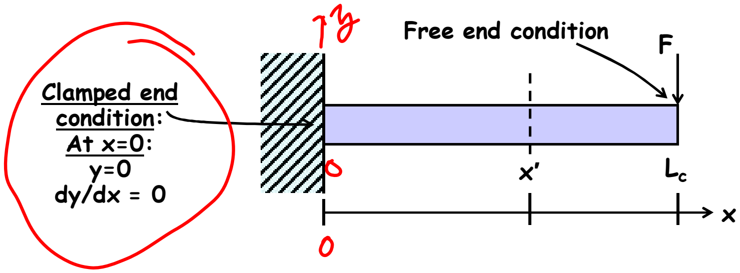
Lecture 13: Beam Bending II

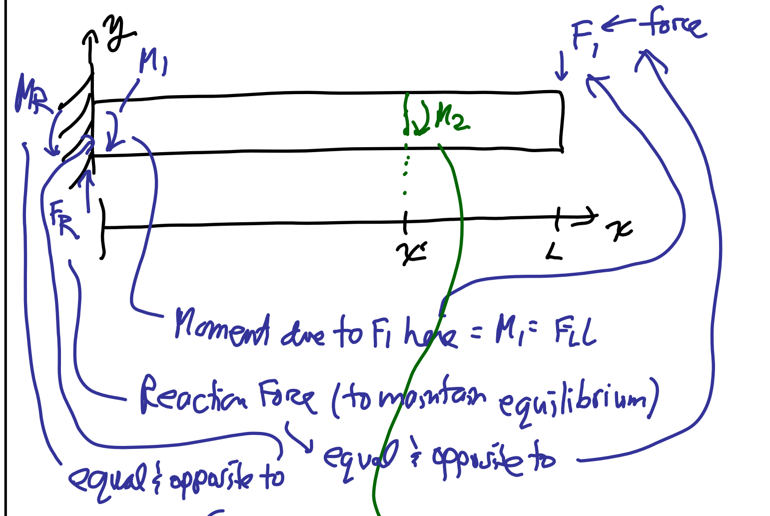
- **Announcements**:
- HW#3 due Wednesday morning
- HW#4 online soon
- Midterm Exam two weeks away, Thursday, March 17, 5:10-7 p.m. in 3107 Etcheverry
- -----------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- -----------------------------------
- **Last Time**:

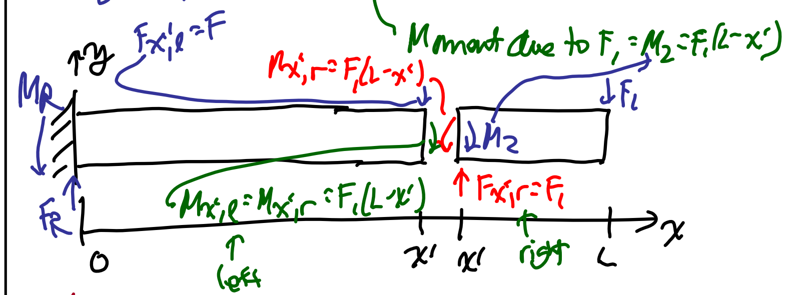
Problem: Bending a Cantilever Beam



- **Objective**: Find relation between tip deflection $y(x=L_c)$ and applied load F
- **Assumptions**:
  1. Tip deflection is small compared with beam length
  2. Plane sections (normal to beam's axis) remain plane and normal during bending, i.e., "pure bending"
  3. Shear stresses are negligible

Forces & Moments





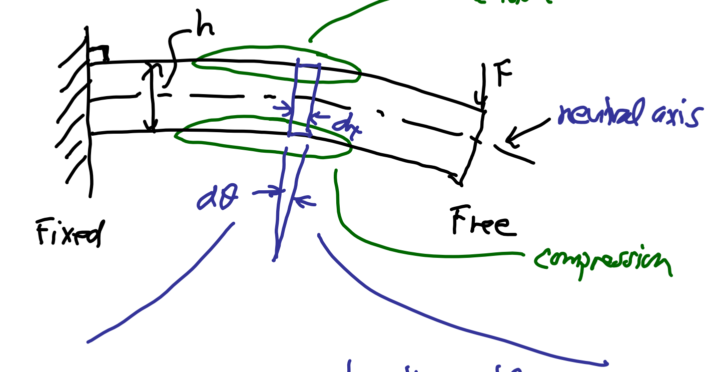
Total Force: $-F_R + F_{x',l} = -F + F = 0$ ✓

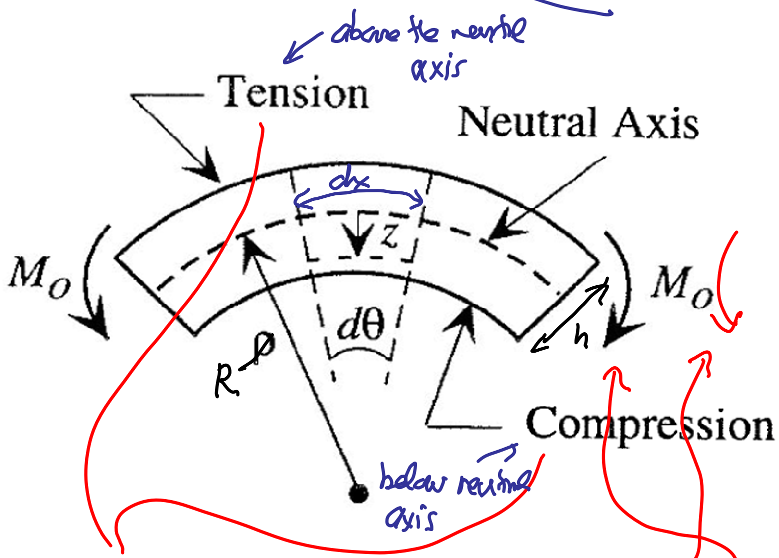
Total Moment: $-M_R + M_{x',l} + x'F_{x',l}$

$= -FL + F(L-x') + x'F = 0$ ✓

Bent Beam (bent by applied force)

⇒ cantilever





Combine to generate forces → moments that against the moment due to force F

↓

This is how stiffness is generated!

↳ k_c → we want this!

Beam Segment in Pure Bending

→ Consider the segment bounded by the dashed lines defining $d\theta$

At $z=0$: neutral axis → segment length $= dx = R\,d\theta$ (1)

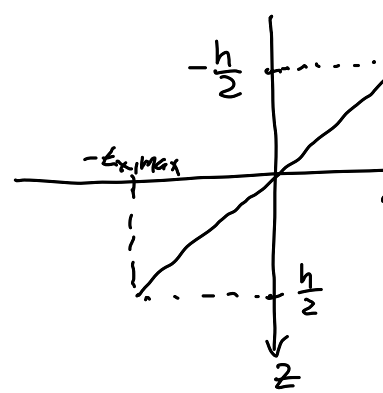
At any z: segment length $= dL = (R-z)\,d\theta$ (2)

Combine (1) & (2): $dL = dx - z\,d\theta = dx - \frac{z}{R}dx$

Thus, the axial strain @ z: $\epsilon_x = \frac{dL-dx}{dx} = -\frac{z}{R}$

$$\boxed{\epsilon_x = -\frac{z}{R}}$$

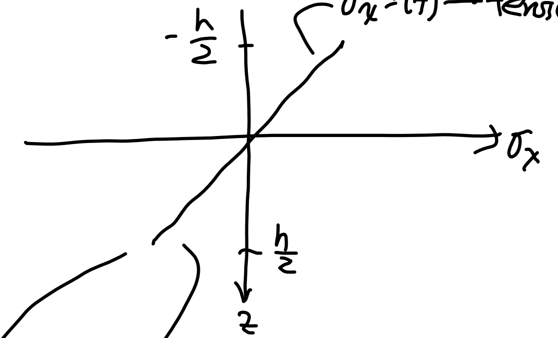
Thus, the strain varies linearly along beam thickness:



$\epsilon_{x,max} = \frac{h/2}{R}$

Of course, there is a corresponding axial stress:

$$\sigma_x = \epsilon_x E = \boxed{-\frac{zE}{R} = \sigma_x}$$

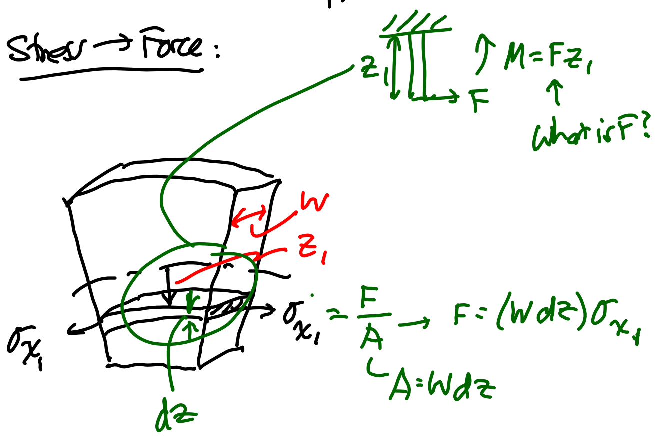


$\sigma_x = (+) \longrightarrow$ tension

$\sigma_x = (-) \longrightarrow$ compression

This gradient of stress generates a bending moment! $\longrightarrow$ in response to the original applied moment (from F)

Stress $\to$ Force:



$\sigma_{x_1} = \frac{F}{A} \to F = (W dz)\sigma_{x_1}$

$A = W dz$

$\Rightarrow$ integrate stress through thickness of the beam:

$$M = \int_{-\frac{h}{2}}^{\frac{h}{2}} [\underbrace{(W dz)\sigma_x}_{\text{force}}] \cdot z$$

$$= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{EWz^2}{R} dz \Rightarrow \boxed{M = -\left(\frac{1}{12}Wh^3\right)\frac{E}{R}}$$

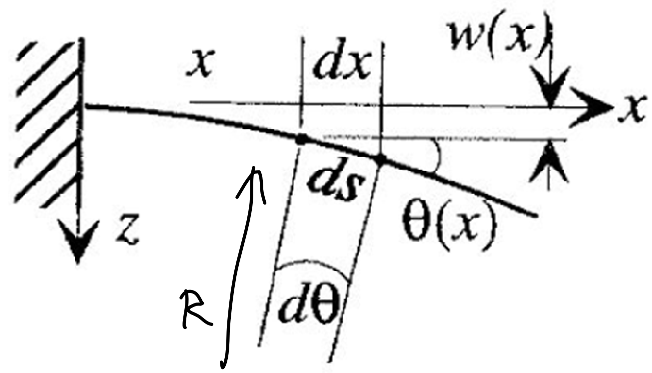
$\left[\sigma_x = -\frac{zE}{R}\right]$

$\frac{1}{12}Wh^3 = I \triangleq$ Moment of Inertia

$$\boxed{\frac{1}{R} = -\frac{M}{EI}}$$

Note: (+) radius of curvature $\to$ (−) internal bending moment

Differential Equation for Beam Bending



Write out some geometric relationships:
(then use small approximations)

$$\cos\theta = \frac{dx}{ds} \rightarrow ds = \frac{dx}{\cos\theta} \longrightarrow ds \approx dx$$

$$\tan\theta = \frac{dw}{dx} = \text{slope of the beam @ this pt.} \longrightarrow \theta \approx \frac{dw}{dx} \quad (1)$$

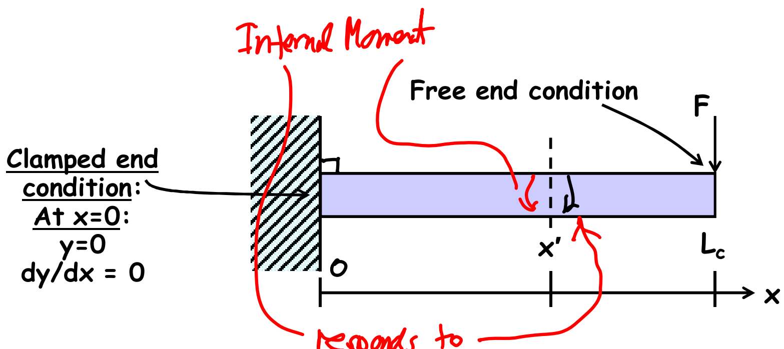
$$ds = R\,d\theta \rightarrow \frac{1}{R} = \frac{d\theta}{ds} \longrightarrow \frac{1}{R} = \frac{d\theta}{dx} \quad (2)$$

Inserting (1) into (2):

$$\boxed{\frac{1}{R} = \frac{d^2w}{dx^2} = -\frac{M}{EI}}$$

Diff. Eq. for Small Angle Beam Bending

Cantilever Beam w/ Concentrated Load



Internal Moment @ position x: $M = -F(L-x)$

Thus: $$\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

w/ $\begin{cases} \text{Clamped End B.C.: } w(x=0)=0,\ \frac{dw}{dx}(x=0)=0 \\ \text{Free End B.C.: none} \end{cases}$

Solve for w:

⇒ use Laplace; or a trial solution:

$w = A + Bx + Cx^2 + Dx^3$, then apply B.C.'s

$$\boxed{w = \frac{FL}{2EI}x^2\left(1 - \frac{x}{3L}\right)}$$

Deflection @ x due to a point load F applied @ $x=L$

Maximum Deflection → occurs @ $x=L$:

$$w_{max} = \left(\frac{L^3}{3EI}\right)F \rightarrow F = \left(\frac{3EI}{L^3}\right)w(x=L)$$

$$= k_c w(x=L)$$

stiffness $\triangleq k_c$

where $k_c = \frac{3EI}{L^3}$

$$\left[I = \frac{1}{12}Wh^3\right] \Rightarrow \boxed{k_c = \frac{1}{4}EW\frac{h^3}{L^3}}$$

Ex. $L = 100\mu m, W = 2\mu m, h = 2\mu m$

polysilicon → $E = 150\,GPa$

$$k_c = \frac{1}{4}(150G)(2\mu)\left(\frac{2\mu}{100\mu}\right)^3 = \underline{0.6\,N/m}$$

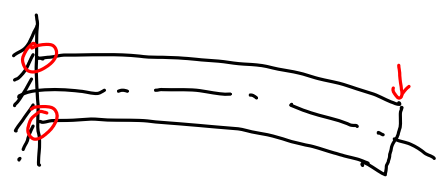
Maximum Stress in a Bent Cantilever

From before, the radius of curvature is given by

$$\frac{1}{R} = \frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

$\Rightarrow \frac{1}{R}$ is maximized (i.e., R is minimized) when $x = 0$:

$$[x=0] \Rightarrow \frac{1}{R} = \frac{d^2w}{dx^2} = \frac{FL}{EI}$$



Strain is maximized: (@ $x=0$)

① At top surface → tensile

② At bottom surface → compressive

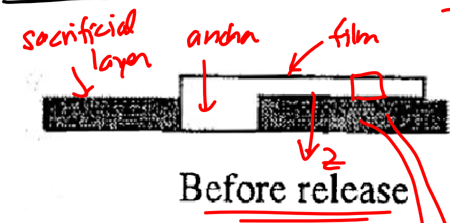
$$\epsilon_{max} = \frac{z}{R} = \frac{h}{2}\frac{1}{R} = \boxed{\frac{h}{2}\frac{FL}{EI} = \epsilon_{max}}$$

$$[I = \tfrac{1}{12}Wh^3] \Rightarrow \epsilon_{max} = \frac{\cancel{h}}{2}\frac{FL}{EI}\left(\frac{12}{Wh^{\cancel{3}\,2}}\right) = \frac{6L}{EWh^2}F \quad *$$

$$* \rightarrow \boxed{\sigma_{max} = \epsilon_{max}E = \frac{6L}{Wh^2}F}$$

(maximum stress in a bent cantilever subjected to a force F at its tip)

Stress Gradients in Cantilevers



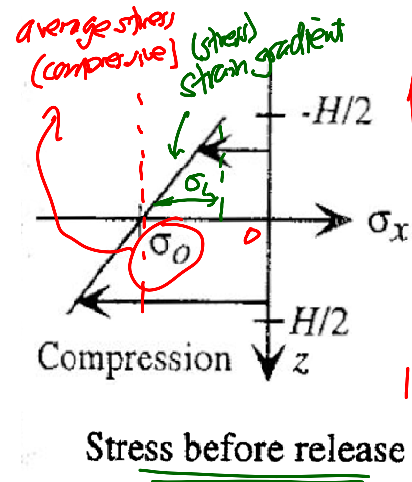
Process Flow:

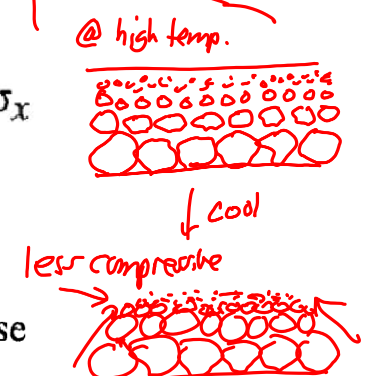
① Deposit film @ high temp

② Cool it down

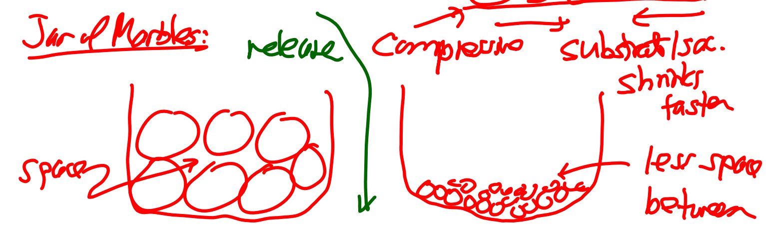
⋮

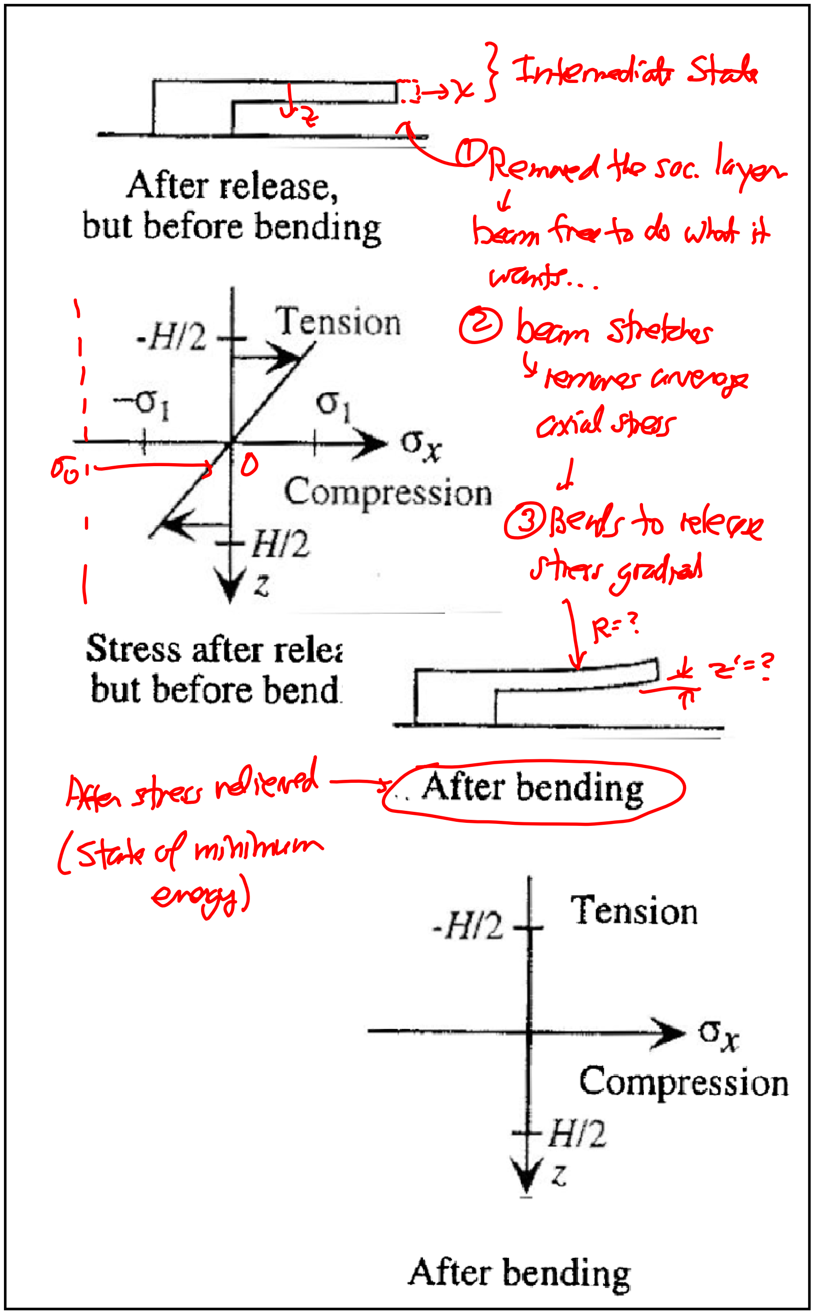
③ Release.





Jar of Marbles:




} Intermediate State
x
z
① Removed the sac. layer
beam free to do what it wants...
After release,
but before bending
② beam stretches
removes average axial stress
③ Bends to release stress gradient
-H/2
Tension
$-\sigma_1$
σ_1
σ_x
σ_0
0
Compression
H/2
z
R=?
z'=?
Stress after relea
but before bend
After stress relieved
(State of minimum energy)
After bending
-H/2
Tension
σ_x
Compression
H/2
z
After bending

