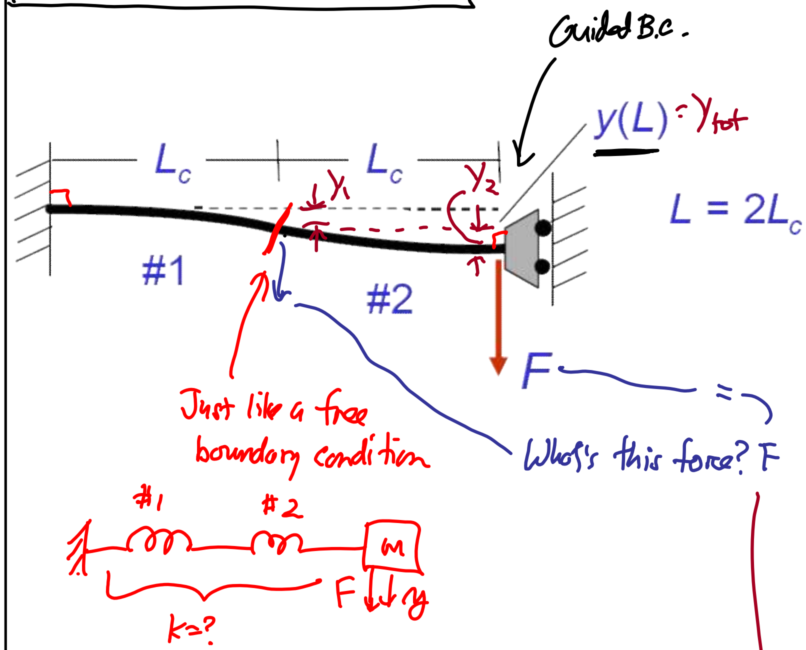
Lecture 15: Beam Combos II

- **Announcements**:
- HW#4 online and due Monday, March 14, 5 p.m.
- Midterm Exam 1.5 weeks away, Thursday, March 17, 5:10-7 p.m. in 3107 Etcheverry
- -------------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients for folded-beam devices
- -------------------------------------
- **Last Time**:
- Analyzing folded-beam suspensions

Series Combination of Springs



Series: $y_{tot} = y_1 + y_2$ } To tell if springs are in series: $y_{tot} > y_1$ ✓ $y_{tot} > y_2$ ✓

$$y_{tot} = y(L) = 2\underbrace{y(L_c)}_{y=f(L_c)} = 2\left(\frac{F}{k_c}\right) = F\left(\frac{1}{k_c} + \frac{1}{k_c}\right) = \frac{F}{k}$$

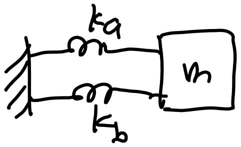
stiffness of a cantilever

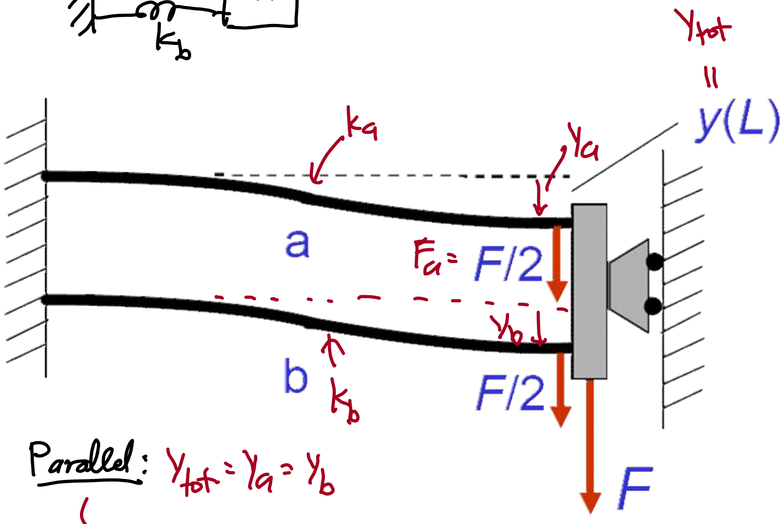
stiffness of whole thing

$$\therefore \boxed{\frac{1}{k} = \frac{1}{k_c} + \frac{1}{k_c}} \rightarrow \boxed{k = k_c \| k_c}$$

Definition for "$\|$" $\left\{ A \| B = \frac{1}{\frac{1}{A} + \frac{1}{B}} = \frac{AB}{A+B} \right.$

Parallel Combination of Springs





Parallel: $Y_{tot} = Y_a = Y_b$

Same displacement of each constituent spring as the total displacement

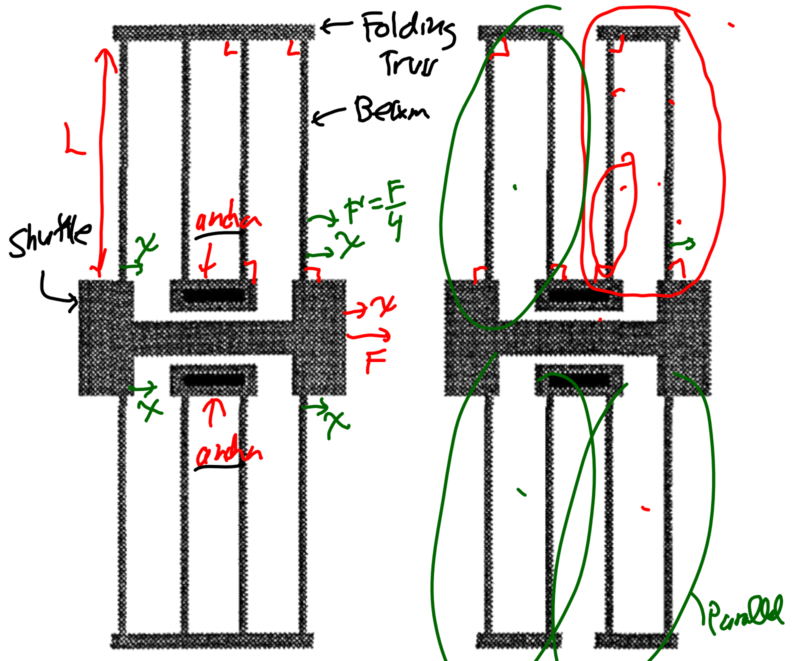
$$y(L) = Y_{tot} = \frac{F}{k} = \frac{\overset{F/2}{F_a}}{k_a} = \frac{\overset{F/2}{F_b}}{k_b} = \left(\frac{F}{2}\right)\left(\frac{1}{k_a}\right)$$

of the whole thing

$\therefore \boxed{k = 2k_a}$

In general: $\boxed{k_{tot} = k_a + k_b}$

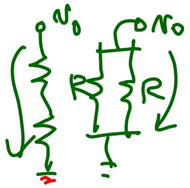
For EE's: springs combine like capacitors

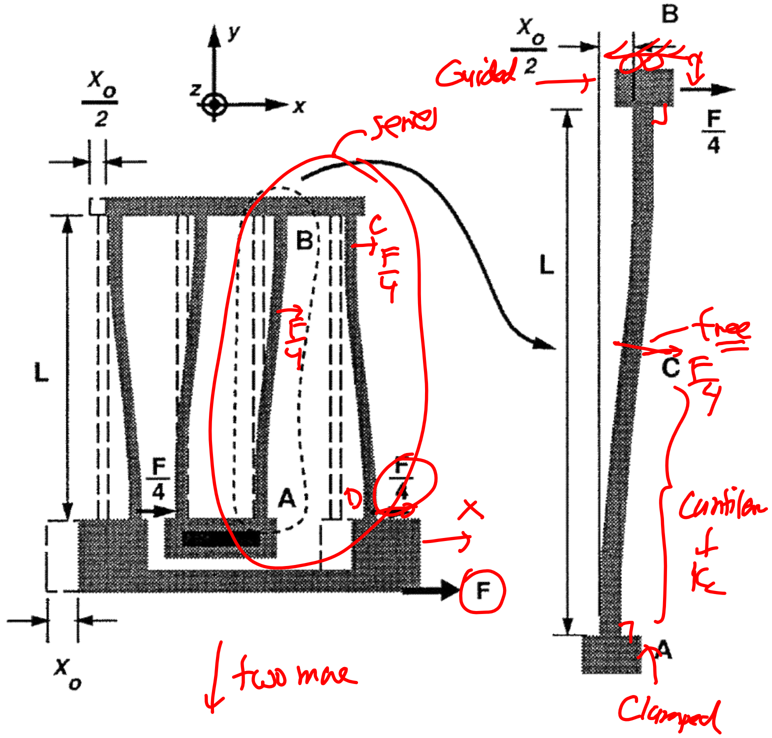


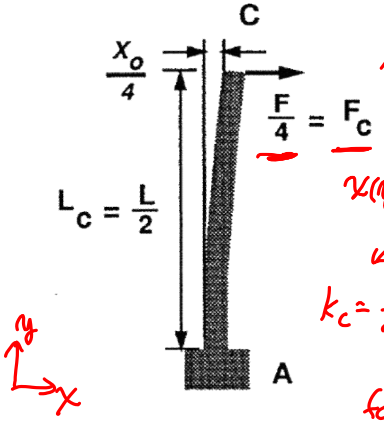
(a) Inner fold, continuous truss

(b) Inner fold, discontinuous truss

Want $x = f(F) = \frac{F}{k_{tot}}$







$$x(y) = \frac{F_c L_c}{2EI_z} y^2 \left(1 - \frac{y}{3L_c}\right)$$

↓

$$x(y) = \frac{F_c y^2}{6EI_z}(3L_c - y)$$

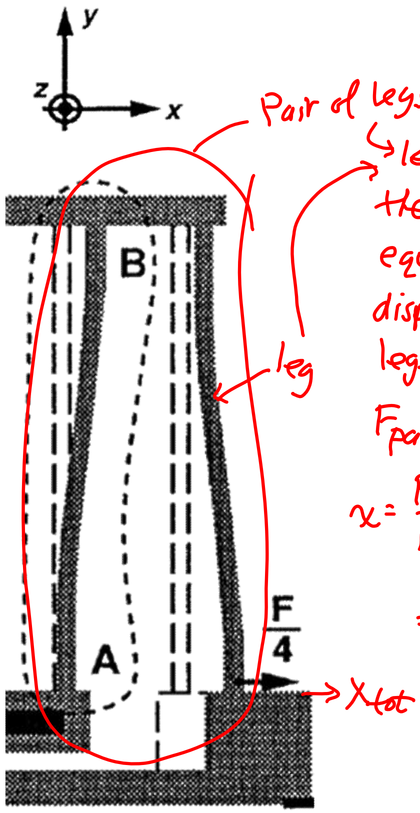
$$k_c = \underset{\text{fcn}}{\frac{F_c}{x(L_c)}} = \frac{3E\overset{\frac{1}{12}hW^3}{I_z}}{L_c^3} = \frac{1}{4}E\underset{\text{thickness}}{h}\frac{W^3}{L_c^3}$$

⇒ insert $L_c = \frac{L}{2}$:

$$\boxed{k_c = \frac{3EI_z}{(L/2)^3} = \frac{24EI_z}{L^3}} = \frac{1}{4}Eh\left(\frac{W}{L_c}\right)^3 \quad \nwarrow L_c = \frac{L}{2}$$

Stiffness of cantilever of length L_c @ $y = L_c$

Full Beam Length



↳ legs in series, because the total displacement X_{tot} equals the sum of the displacements of the individual legs

$F_{pair} = \frac{F}{4}$ ← applied to the shuttle

$$x = \frac{F_{pair}}{k_{pair}} = \frac{F_{pair}}{(k_{leg} \| k_{leg})}$$

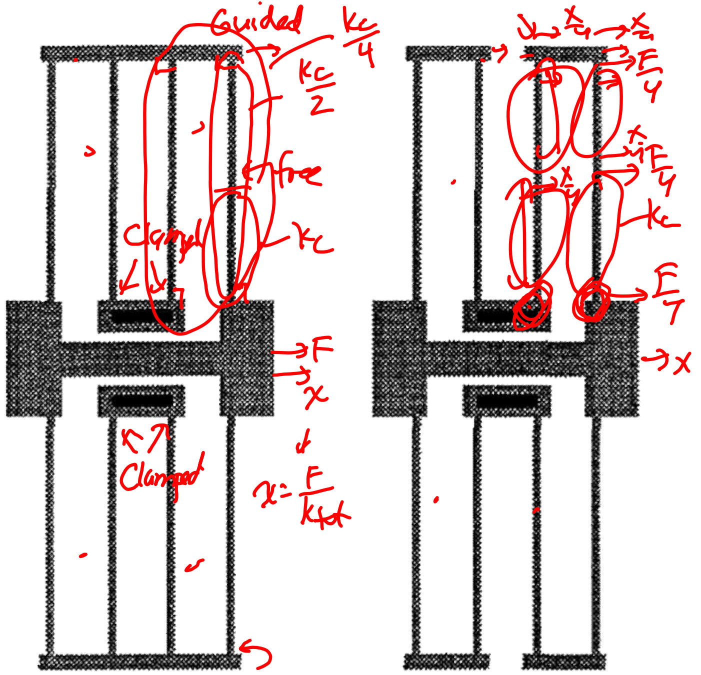
$$= \frac{F}{4}\left(\frac{1}{k_{leg}} + \frac{1}{k_{leg}}\right)$$

From before: $k_{leg} = k_c \| k_c = \frac{k_c}{2}$

Thus: $x = \frac{F}{4}\left(\frac{2}{k_c} + \frac{2}{k_c}\right) = \frac{F}{k_c} = \frac{F}{k_{tot}}$

$$\boxed{k_{tot} = k_c = \frac{24EI_z}{L^3}}$$

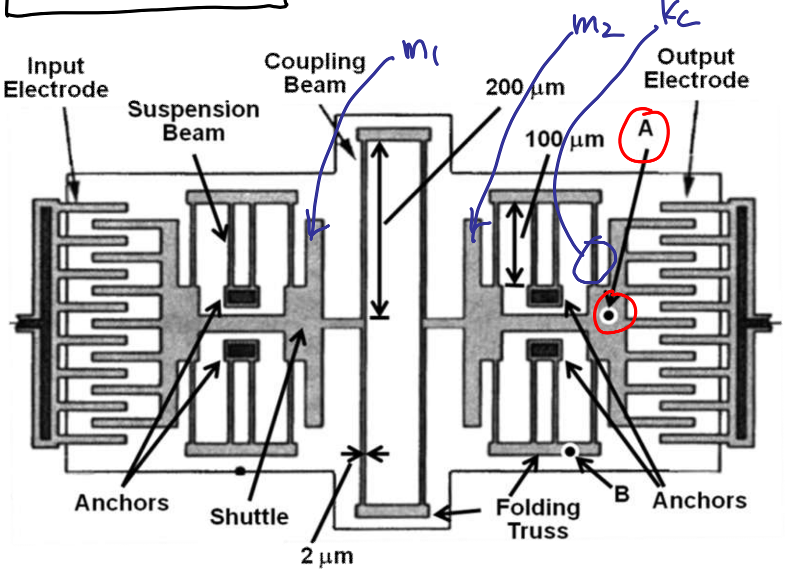
Better Way to Do It → Just consider stiffnesses



(a) Inner fold, continuous truss (b) Inner fold, discontinuous truss

$k_{tot} = 4\left(\frac{k_c}{4}\right) = k_c$ ✓

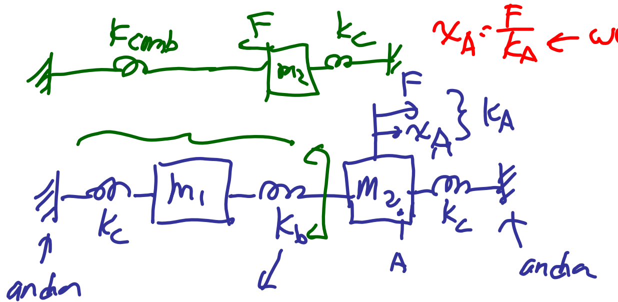
Micromechanical Filter



⇒ Find the stiffness at point A
(shuttles are rigid)
(folding trusses " ")

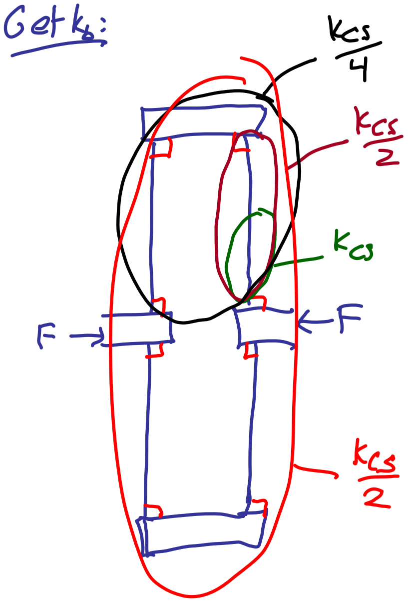
↳ apply F @ A → what is x_A?
$k_A \triangleq$ stiffness @ point A

$x_A = \frac{F}{k_A}$ ← want this



$k_b \overset{?}{=} \rightarrow \frac{k_{cs}}{2}$

Get k_b:

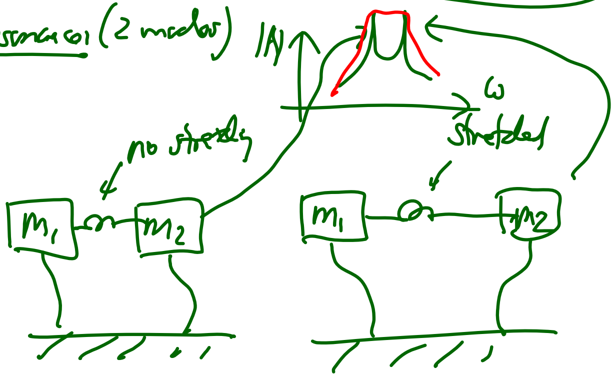


$\therefore k_A = k_c + k_{comb}$

$k_A = k_c + k_c \| \frac{k_{cs}}{2}$ where $k_c = \frac{24EI_z}{L_r^3}$

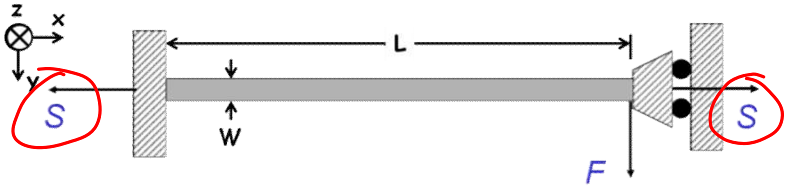
$k_{cs} = \frac{24EI_z}{L_{cs}^3}$

Resonance (2 modes)



Tensioned Spring (Non-Ideality)

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L

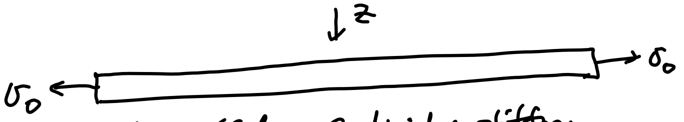


Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4y}{dx^4} - \underset{\text{Axial Load}}{S} \frac{d^2y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

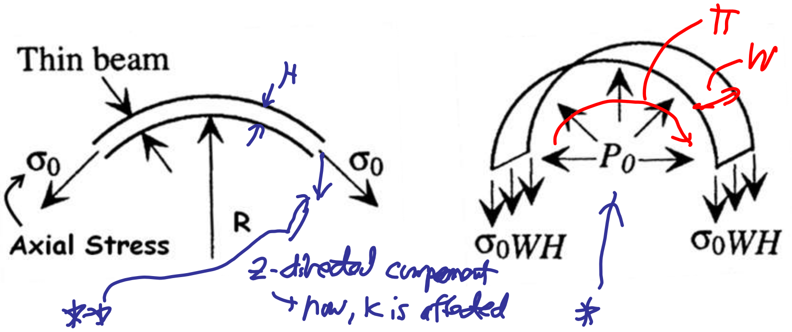
Heuristic Derivation for the Euler Beam Equation

Consider first a straight beam under axial stress:



→ no effect on z-directed stiffness when the beam is straight

... but when the beam is bent:



z-directed component
→ now, k is affected

* ↳ Upward pressure P_o to counteract the

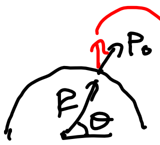
** downward force from ↳ to keep everything in static equilibrium

For ease of analysis:

Assume the beam is bent to an angle π

↳ Downward vertical force: $2\sigma_o WH$

Upward Force due to P_o:



$$P_{uy}(\theta) = P_o \sin\theta$$

$$F_u = \int_0^{\pi} (P_o \sin\theta)\, W (R\, d\theta)$$

$$= -P_o W R \cos\theta \Big|_0^{\pi}$$

$$= 2RWP_o$$

[Equilibrium] $\Rightarrow 2RWP_o = 2\sigma_o WH \rightarrow P_o = \dfrac{\sigma_o H}{R}$

$$\left[q_o = \frac{\text{beam load}}{\text{unit length}} = P_o W, \quad \frac{1}{R} = \frac{d^2 w}{dx^2} \right]$$

beam displacement in z-direction

$$q_o = \sigma_o WH \frac{d^2 w}{dx^2}$$

generalizes to the case of smaller displacement angles

Using the differential beam bending equation

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \xrightarrow[???]{} \frac{d^4 w}{dx^4} = \frac{q}{EI}$$

load / unit length

↓ *

* ↳ Relationships Between Forces on a Fully Loaded Differential Beam Element

q = force/unit length